The Riemann hypothesis is undecidable in arithmetic (vi)

by

Dr P Braun

Notation, functions, conventions

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_{p\ prime}(1-\frac{1}{p^s})^{-1}\ ,\quad P(s) = \sum_{p\ prime}\frac{1}{p^s}\ ,\quad \log\zeta(s) = \sum_{k\geq 1}\frac{1}{k}P(ks)$$

where s=σ+ it and Re{s} is σ>1.
The quasi -Riemann hypothesis RH(θ) is the proposition that ζ(s)≠0 for σ> θ (1/2< θ <1).
The von Mangoldt function *Λ*(n)=log p if n is a positive integer power of p and is otherwise 0.
The Mobius function μ(1)= 1 , μ(n)=0 if n is divisible by a square and otherwise μ(n)= -1.
M(x) denotes the sum of the μ(n) up to [x].
The number Θ is the reserved value for lub {θ: ζ(s)≠0 for σ> θ}.
For the most part of this discussion we assume 1/2≤Θ<1 and the case Θ =1 is discussed separately nearer the conclusion.
The critical strip is the region 0≤σ≤1.
{n} denotes the ordered sequence of natural numbers and {p} denotes the ordered sequence of prime numbers.
In the context of Dirichlet series,

$$\mathrm{Cosum}_x\big(f(s)\big) = \sum_{n\leq x} a(n), \qquad \text{where} \quad f(s) = \sum_{n\geq 1}\frac{a(n)}{n^s}\,.$$

The symbol ≡ denotes logical equivalence of propositions.

Mentions of big O(x), little o(x), and variations on Ω(x) all assume 'as x→∞'.

The long run primes are the set of primes implicit in the convergence of the Dirichlet series representation of P(σ) and P(s) in σ>1.

Background

We use a distinction in the context of studying the ordered prime distribution between the finite primes, the primes referenced by P(σ), and the primes referenced by P(s). We have a correspondence here with work in the rational field of simple arithmetic, the elementary methods and analytical work in the real field, and the analytic work in the field of complex numbers. Through the series of constructions - N→Z→Q→R→C and forward mappings, we may study the prime numbers in the unified structure of complex analysis. The full chain of the construction process, preserves the binary operations and algebraic patterns within the earlier structures.

Summary
In the theory of ζ(s) developed in the complex number field, it is in the nature of the analytic continuation of P(s) and P'(s) in ½<σ<1 that we find the resolution of the problem of the Riemann hypothesis as 'undecidable'.
In outcome terms, the continuations of P(s) and P'(s) contribute to the abstract theory but not to the more practical problem of establishing the existence of a zero of ζ(s) in ½<σ<1. In this region, ζ(s)=0 at a point, if and only if P(s) has a singularity at that point and that is all. There is no known algebraic/analytic connection between P'(s) and ζ(s) other than their poles at s=1.
This crucial inhibitor to locating a pathway to verify a true/false outcome to RH(θ) is discussed in sections 3 and 4.
Titchmarsh [2] Chapter (IX) covers the basic theory of Dirichlet series.

An analytic continuation of P(s) using the inductively driven side of known properties of ζ(s) at regular points in 0< θ <1, is

$$P(s) = \sum_{n\geq 1} \frac{\mu(n)}{n} \log \zeta(ns) \qquad (\sigma > 0),$$

(Titchmarsh [1] Chapter 9 and Appendix 1)

In the halfplane σ>1/2, a simpler continuation with the same basic algebraic characteristics

$$P(s) = \log \zeta(s) - \sum_{k\geq 2} \frac{1}{k} P(ks) \qquad (\sigma > 1/2)$$

is preferred here and discussed in Section 4.

In this latter half plane, P(s) singular is equivalent to ζ(s)=0.

A well-known collection of equivalences to RH(θ) is shown to be undecidable and without numerical contradiction in complex variable theory.
Namely, in ½< θ <1,

$$RH(\theta) \equiv \sum_{p\leq x} \log p = x + O\left(x^{\theta+\varepsilon}\right) \text{ as } x \to \infty \ldots\ldots\ldots\ldots (1),$$

(Ingham Chapter (IV), Davenport [1] Chapter 18).

In terms of Dirichlet series this collection translates to the propositions that

$$RH(\theta) \equiv P'(s) + \zeta(s) \text{ is convergent for } \sigma > \theta \ (½ < \theta < 1) \ldots\ldots..(2).$$

If such a proposition is proven for a fixed value of θ, then we have a regular continuation of P'(s) to σ> θ.
This is equivalent to ζ(s)≠0 in σ> θ.

The von Mangoldt formula for the sum of the primes logarithms provides a link to the long run primes.

The formula may be expressed in the truncated form

$$\sum_{\substack{p\ \mathrm{prime} \\ p\leq x}} \log p = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + O\left(x^{\left(\frac{1}{2}\right)+\varepsilon}\right) \text{ as } x \to \infty$$

where the summation is in the grouping order of increasing $|\mathrm{Im}(\rho)|$ over the non-trivial zeros of ζ(s) in σ>1/2,
Edwards [1] Chapter 3, Davenport [1] Chapter 17.
Consequently, (1) and (2) are inextricably linked to the complex number zeros of ζ(s) in ½<σ<1.
The nature of the analytic continuations of P(s) and P'(s) are used to deduce that the RH(θ) propositions are undecidable in ½< θ<1.
i.e. the algebraic nature of P(s) and P'(s) in continuation prevent the theory from being able to identify singularities of P(s) in ½< θ<1.

Section1

Historical limitations of real analysis

In real analysis, the Dirichlet series for P(σ) and P'(σ), convergent for σ>1, do not yield much information about the values of the long run primes. A certain kind of fusion in real analysis methods between {p} and {n} enables an estimate for counting the primes in the prime number theorem

$$\sum_{\substack{p\leq x \\ p\ \mathrm{prime}}} 1 \quad = \pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad \text{as } x \to \infty.$$

The mathematical activity in the history of the problem provides support to the observation that real analysis, 'only just' gets passed the leading asymptotic estimates in the sense that

$$M(x) = 0x + o(x^1), \quad \sum_{\substack{p\ \mathrm{prime} \\ p\leq x}} \log p = 1x + o(x^1) \text{ and } \sum_{\substack{p\ \mathrm{prime} \\ p\leq x}} 1 = \sum_{n\leq x} \frac{1}{\log(n)}) + o(x^1).$$

In real and complex analysis, there are no known results in ½< θ<1 for any of the equivalent

$$M(x) = 0x + O\left(x^{\theta+\varepsilon}\right), \sum_{\substack{p\ \mathrm{prime} \\ p\leq x}} \log p = x + O\left(x^{\theta+\varepsilon}\right), \quad \sum_{\substack{p\ \mathrm{prime} \\ p\leq x}} 1 = \sum_{n\leq x} \frac{1}{\log(n)}) + O\left(x^{\theta+\varepsilon}\right).$$

These propositions require the ability to recognise actual singularities of P(s) and P'(s), should such thing exist in ½< θ < 1. The early estimates use the singularity at s=1 of either P(s) or P'(s) as key to the first term asymptotic estimates above. Any advances on these arithmetic propositions in half planes would prove P(s) and P'(s) regular for the discovered θ < 1 and hence we would have ζ(s)≠0 in the wider half plane than σ≥ θ .

The complete von Mangoldt identity is a beautiful end point in the theory, an intellectual monument, which in itself is a primary key to understanding the nature of the Riemann hypothesis and the error term in relevant asymptotic estimates involving the long run prime numbers. That is, the existence of some or all of the complex zeros of ζ(s) in the critical strip, need to be taken into account in the 'remainder' terms relating to θ type estimates.

Section 2

Hunting down the long run distribution of the values of primes.

Each extension field places its own conditions on the values of the prime numbers in terms of their distribution and what can be proved. The divergence of P(σ) at σ =1 places a weak density condition on the long run of primes in real analysis.

In locating properties of the long run of primes with reference to their values, we are not discovering properties of the primes per se, but properties of the primes within certain defined structures up to the field of complex numbers. These properties cannot be contradicted in elementary rational arithmetic in light of the established construction process. We explain why a true/false resolution for any one value of θ in the equivalences under discussion is asking too much from the complex analysis examining the remainder term in the long run values of the primes. These values are well and truly sheltered in P(s) and P'(s) to prevent resolution of the θ problem from the inductive gaze of the natural numbers and thus ensure the problem is undecidable. The standoff is seen in the line of convergence of the function P'(s)+ζ(s) rearranged as a Dirichlet series, between the prime sequence {p} and the natural number sequence {n}.

The halting of elementary methods at the line σ=1 signals the possibility that, these two functions have nothing in common to work with, except their cancelling simple poles at s=1. with opposite signs. This blocks the proof for an improved half plane of convergence. The purpose of the remaining discussion is to explain causal elements to match these observations.

Section 3

A pathway to undecidability of RH(θ) is now available in complex variable theory

Although, we only have one lot of prime values in arithmetic as defined by their order in the natural numbers, mapping the reals into the complex numbers, produces a significant jump up from the conditions on the primes in the ancient real variable Euler product for ζ(σ). This was hoped to provide wider opportunities for significant advances in the nature of the log run distribution of the prime values, when identified in the analytic continuation of the Euler product using ζ(s). i.e. the increased scope in moving to

$$\zeta(s) = \prod_{\mathrm{p\ prime}} (1 - \frac{1}{\mathrm{p}^{s}})^{-1} \quad \text{from } \zeta(\sigma) = \prod_{\mathrm{p\ prime}} (1 - \frac{1}{\mathrm{p}^{\sigma}})^{-1} \quad .$$

The wider field of the complex numbers may impose more constraints in terms of prime values than the field of real numbers in the sense of producing stronger results.

It turns out though that the new environment is both friendly and hostile.

An example is the result that the oscillatory nature in the values of remainder term for M(x) and the amplitudes, positive and negative, are directly related to the actual value of θ in ½≤ θ <1 in the theory. A very friendly result from the power of complex analysis. However, the undecidability of RH(θ) may be viewed as a hostile outcome to the true/false mathematician respecting past efforts.

The point s=1 and the line σ=1 provide the cross over from real analysis to a domain in which the values of the primes are possibly more accessible in terms of finding a half plane in which RH(θ) is true. The hard-fought gains in real analysis over many decades exhibit some elasticity for tweaking the estimates for the remainder terms involved in the RH(θ) above. They rely on what might be called inductively driven methods using the inductive side of ζ(s) theory and/or sieve methods. However, the values of the long run primes are essentially already in prison in P(σ) and it turns out, are at least as restricted in the higher security prison of P(s).

The difficulty of quantifying the remainder terms of θ for the long run primes is reflected in

$$\mathrm{RH}(\theta) \equiv \sum_{N\le p\le x} \log p = x + O\left(x^{\theta+\varepsilon}\right) \text{ as } x \to \infty \qquad \left(\frac{1}{2} < \theta < 1\right)$$

for any finite fixed N, no matter how large. The needs of the equivalences may ignore the finite counting primes in the summation.
i.e. the primes of interest are those which are out of reach of visualisation.
We may be tempted to think of these primes as foot loose and fancy free and not revealing the values of their finite counterparts.
This fits in nicely with the 'virtual' nature of P'(s) being unable to contribute to the numerical requirements of RH(θ), relative to the theoretical equivalences as discussed in Section 4.
We may hope the weighted count of the {log p} sequence smooths the distribution of the values of the p values to the point where the remainder term in the equivalences in able to find a θ <1 but it is still a long run count like the prime count of the prime number theorem.

We recall from section 1 that

$$\sum_{\substack{p \text{ prime} \\ p\le x}} \log p = x + O\left(x^{\theta+\varepsilon}\right) \equiv \sum_{\substack{p \text{ prime} \\ p\le x}} 1 = \sum_{n\le x} \frac{1}{\log(n)}) + O\left(x^{\theta+\varepsilon}\right) \equiv \mathrm{RH}(\theta).$$

In section 4, amongst other things, we see that remainder term in each of these propositions turns out to be inappropriate in the language and rules of the theory.

In the following section we argue that establishing any half plane of convergence of P'(s)+ζ(s) is undecidable.

Section 4

The problem of classical continuations of P(s) and P'(s) in ½<σ<1

We now place the Euler product for ζ(s) at the gateway to examining possible vertical half-planes of convergence in which ζ(s) is non-zero.

$$\zeta(s) = \sum_{n\ge 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1} \qquad (\sigma > 1).$$

As noted above, this truth does not require much of the values of long run primes, whereas all is known about the values of natural numbers with the rule n→n+1.

This relationship started out from the intuitive thought that if the RHS is multiplied out and ordered in terms of values, without any s, there is a unique representation of each natural number in terms of their prime decomposition.

With the arrivals of real analysis and complex variable theory, the validity of the equality was established. The intuitive 'global' relationship then had a well- defined place in the language of analysis. What we are missing is another global connection between {n} the long run primes {p}.

When we get to the RH(θ) equivalences involving the values in the {log p} sequence in the long run, we go well beyond the needs required in validating the Euler product formula. The analytic requirement on the values of the primes in the Euler product (Titchmarsh [1]) is so weak it seems quite remarkable that the prime number theorem ever came into being. The association between of ζ(s) and P'(s) in the complex variable domain has the von Mangoldt formula as a reflexion of the complexity of the relationship, and the undecidability of the RH(θ) as a reflexion of the tenuous nature of the connection. We show that the two players are held together in $\frac{1}{2}<\sigma<1$ only by the logical fixture that ζ(s)=0 if and only if P(s) is singular.

Indeed, each of the functions P(s) and P'(s) has a natural defined domain of analyticity in $\sigma>1$ in terms of Dirichlet series and a defined analytic continuation to $\sigma>0$. The capacity of each form of these continuations to aid in identifying poles of P'(s) in $\frac{1}{2}<\sigma<1$ is blocked by the peculiar nature of the algebraic structure of the continuation

$$P(s) = \log\zeta(s) - \sum_{k\geq 2}\frac{1}{k}P(ks) \qquad (\sigma > 1/2) \ \dots\dots\dots (3).$$

The RHS above is regular for $\sigma>1/2$ avoiding possible zeros and the pole of ζ(s).
The result follows from the estimate
$|P(ns)| = O(1/2^{n\sigma})$ as $n \to \infty$ (Titchmarsh [1] and Appendix 1).
This is sufficient to justify the continuations.

A derivation of the analytic continuation of P(s) to $\sigma>0$ is discussed in the form

$$P(s) = \sum_{n\geq 1}\frac{\mu(n)}{n}\log\zeta(ns) \quad \text{(Titchmarsh [1] Chapter 9.)}$$

On the surface it looks as if we have engaged a very inductively driven function ζ(s) to give P(s) some additive type credibility but it is log ζ(s) and the Euler product which creates the illusion of an additive and multiplicative connection in this problem. The deflection from a possible further significant connection of a global type nature between {p} and {n} derives from the self-referential nature of the continuation.
It is helpful to write the continuation (3) in terms of P(s) in brackets to see what is going on in $1/2<\sigma<1$. This is an algebraic identity at regular points which has an analytic interpretation. Most importantly, P(s) here defines its own continuation in the identity to $\sigma>1/2$

$$P(s) = \left\{P(s) + \frac{1}{2}P(2s) + \frac{1}{3}P(3s) + \cdots\right\} - \left\{\frac{1}{2}P(2s) + \frac{1}{3}P(3s) + \cdots\right\} \qquad \dots\dots(4).$$

Worst still, to obtain the value of P(s), regular or singular, we need to assume the imbedded value of P(s).
All we can deduce in $1/2<\sigma<1$ is that P(s) has a singularity at a point if and if ζ(s)=0 at that point.
What we are missing here is the analytic continuation of P(s) in $\sigma>1/2$ connected to inductively driven entities.

It looks more like a circus trick generated in the prime number value prisons of P(s), P(2s), P(3s), trapped in a P(s) driven universe with no substantial connection to the inductive world of the additive mathematics of the natural numbers. Importantly, we cannot identify a singularity of P(s) in ½$< \theta < 1$ by numerical calculation from the component parts in (4).
A similar result follows for P'(s) (Titchmarsh [1] and Appendix 1).

We call this type of isolated continuation – virtual - in the context of trying to link additive and multiplicative workings in the theory of prime number distribution in complex variable theory.
The P(s) and P'(s) do not have accessible values to use in a theoretical situation and any use of them in developing theoretic results will be conditional on the values of P(s), P'(s) being regular and/or accommodating singularities.
It is the rare situation here of the natural numbers examining this aspect of themselves and we see the justification to place a 'persona non-grata' status on this kind of analytic continuation. It looks like a continuation and behaves like a continuation in the theory but it cannot distinguish between regular and singular points when used in the theory.
No convergent value of P(s) can be verified numerically.

Pushing ahead in the established theory, from the equivalences

$$\mathrm{RH}(\theta) \equiv \sum_{p\le x} \log p = x + O\left(x^{\theta+\varepsilon}\right) \text{ as } x \to \infty \ ,$$

and in terms of Dirichlet series

$$\mathrm{RH}(\theta) \equiv \left\{\mathrm{Cosum}_x\left(P'(s) + \zeta(s)\right) = O\left(x^{\theta+\varepsilon}\right)\right\} .$$

We note this would provide a regular continuation of P'(s) in ½$< \theta < 1$ given that $\zeta(s)\neq 0$ on $\sigma=1$.

Here then, quite explicitly, we see a direct confrontation between the multiplicative base and the additive natural numbers in the long run. A genuine impasse. The nearest we can get to a productive conversation in regard to this problem is the Euler product relationship, which is where we started from.

In the theory, the analytic nature of $P'(s) + \zeta(s)$ in the half plane $\sigma>1/2$ cannot distinguish between regular and singular points outside of those in the half plane where $\zeta(s)$ is non-zero because of the virtual nature of P'(s). The poles of P'(s) and $\zeta(s)$ at s=1 cancel and do not come into play in the ordered Dirichlet series.

Any use of the P(s) and P'(s) extensions in theoretical work will of course be valid because it will incorporate working with the possible zeros of $\zeta(s)$ and poles of P'(s).
The structure of the positive argument is to work with things which have been proved and then add on things, which need to be proved for a standard formal argument to occur. However, if one of the necessary things which needs to be proved is itself virtual, the argument does not constitute a true/false 'proof'.
This relationship between P'(s) and $\zeta(s)$ here is virtual because the P'(s) is virtual in this domain. i.e. from the analytic continuation we only get what we started with- (more or less)- P'(s) is singular in this region if and only if $\zeta(s)=0$ at the point.

Section 5
The case Θ=1

This can only occur if there exist increasing real sequences $\{\sigma_n\}$ and $\{t_n\}$ with ($1/2<\sigma_n<1$) and $\zeta(\sigma_n + it_n) = 0$ and $\sigma_n \to 1$ as $n\to\infty$.
A proof of this would contradict RH(θ) as undecidable ½$< \theta < 1$.

Is M(x)=O(√x)?

Some arithmetic propositions follow from this 'undecidability' with the certainty of 'no numerical contradiction' but it appears that the modified Merten's conjecture M(x)=O(√x) cannot be called as undecidable. This may leave the simplicity of the zeros on σ=1/2 an open question.
See Ollyzko A.M. & te Riele H.J.J. [1].

Appendix 1

Proposition1

Let $1 < X_1 < X_2 < X_3 < \cdots$ be monotone increasing natural numbers and complex numbers a_i satisfy $|a_i| \le 1$.
Then

$$\zeta_{X,k}(n\sigma) = \sum_{k\ge 1} \frac{a_k}{X_k^{n\sigma}} = O\left(\frac{1}{X_1^{n\sigma}}\right) \text{ as } n \to \infty \quad \left(\sigma > \frac{1}{2} \text{ and integer } n > 1\right).$$

Proof
It suffices to show

$$\sum_{k\ge 1} \frac{1}{X_k^{n\sigma}} = O\left(\frac{1}{X_1^{n\sigma}}\right) \text{ as } n \to \infty \quad (\sigma > 1).$$

We start with the very worst case which with σ>1 is

$$\zeta(n\sigma) - 1 \quad \le \quad \left\{\frac{1}{2^{n\sigma}} + \frac{1}{2^{n\sigma}}\right\} + \left\{\frac{1}{4^{n\sigma}} + \frac{1}{4^{n\sigma}} + \frac{1}{4^{n\sigma}} + \frac{1}{4^{n\sigma}}\right\} + \\ + \left\{\frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}} + \frac{1}{8^{n\sigma}}\right\} + \cdots.$$

i.e.

$$|\zeta(n\sigma) - 1| \le \sum_{k\ge 1} \frac{1}{2^{k(n\sigma-1)}} \quad \text{as } n \to \infty.$$

For sufficiently large n

$$|\zeta(n\sigma) - 1| \le \frac{1}{2^{n\sigma-1} - 1} = \frac{1}{2^{n\sigma-1}}\{1 - \frac{1}{2^{n\sigma-1}}\}^{-1} = O\left(\frac{1}{2^{n\sigma}}\right) \text{ as } n \to \infty.$$

We may apply the same groupings to the sequence used above for ζ(nσ)-1 on

$$|\zeta_{X,k}(\sigma) - 1| \le \sum_{n\ge 1} \frac{|a_k|}{X_1^n}$$

to get proposition 1.
In particular, with a suitable choice of the $X_i^{n\sigma}$ and the a_i we have

$$\log \zeta(ns) = O\left(\frac{1}{2^{n\sigma}}\right) \text{ as } n \to \infty \text{ and } P(ns) = O\left(\frac{1}{2^{n\sigma}}\right) \text{ as } n \to \infty.$$

References

[1] Davonport. H.
Multiplicative Number Theory, Lectures in Advanced Mathematics, Markham Publishing Company 1967.

[1] Edwards H.M. Riemann's Zeta Function. Dover Publications, Inc. Mineola, New York 2001.

[1] A.O. Gelfond & Yu.V. Linnik.
Translation. Elementary methods in Analytic Number Theory. Rand McNally and Company, Chicago, 1965

[1] Ollyzko A.M. & te Riele H.J.J.
Disproof of Merten's Conjecture.
Journal für die Reine und angewandte Mathematik - 357 in: Periodical 225
(www.dtc.umn.edu/~odlyzko/doc/arch/mertens.disproof.pdf)

[1] Titchmarsh E. C.
The Theory of the Riemann Zeta-function, 2nd edition (Revised by D.R. Heath-Brown) Oxford Science Publications.
[2] Titchmarsh E. C.
The Theory of functions, 2nd edition, Oxford University Press 1979.