Section 1.

Some statements equivalent to the quasi-Riemann hypothesis.

As usual in Number Theory, let s be a complex variable, $\sigma = \mathrm{Re}\, s$, $t = \mathrm{Im}\, s$. Let ζ be Riemann's zeta function, and, for $1 > \sigma_0 \geq \frac{1}{2}$, let $RH(\sigma_0)$ be the statement

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \sigma_0 .$$

We refer to this statement as the 'quasi-Riemann hypothesis'. With our notation, $RH(\frac{1}{2})$ will then signify the Riemann hypothesis proper.

In this and later sections we have occasion to use the following result for expressing a Dirichlet series as an integral. The proof of this result is a simple application of a well-known technique but is included here for the sake of completeness.

Proposition 1.

Let $a : \mathbb{N} \to \mathbb{C}$ satisfy

$$A(x) = \sum_{n \leq x} a(n) = O(x^{\Delta})$$

as $x \to \infty$. Then for $\sigma > \Delta$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}}\, dx .$$

Proof:

For $\sigma > \Delta$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{a(n)}{n^s}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{A(n) - A(n-1)}{n^s} \quad (A(0) = 0)$$

$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} A(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{A(N)}{N^s} \right\}$$

$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} s \int_{n}^{n+1} \frac{A(x)}{x^{s+1}}\, dx + \frac{A(N)}{N^s} \right\}$$

$$= s \int_{1}^{\infty} \frac{A(x)}{x^{s+1}}\, dx,$$

and the function defined by the integral is analytic for $\sigma > \Delta$.

For real κ let

$$S_\kappa(x) = \sum_{n\le x} \lambda(n) n^\kappa, \quad M_\kappa(x) = \sum_{n\le x} \mu(n) n^\kappa \quad ,$$

$$h_\kappa(x) = \sum_{n\le x} \lambda(n) n^{\kappa-1}, \quad g_\kappa(x) = \sum_{n\le x} \mu(n) n^{\kappa-1},$$

$$H_\kappa(x) = \sum_{n\le x} h_\kappa(n), \quad G_\kappa(x) = \sum_{n\le x} g_\kappa(n) \quad ,$$

where λ is Liouville's function, and μ is the Möbius function.

Proposition 2.

Let either $\kappa = -1$ or $\kappa > -\sigma_0$. Then the following statements are equivalent:

(i) $RH(\sigma_0)$,

(ii) $\forall\ \varepsilon > 0,\ S_\kappa(x) = O(x^{\sigma_0+\kappa+\varepsilon})$ as $x \to \infty$,

(iii) $\forall\ \varepsilon > 0,\ H_{\kappa+1}(x) = O(x^{\sigma_0+1+\kappa+\varepsilon})$ as $x \to \infty$,

(iv) $\forall\ \varepsilon > 0,\ M_\kappa(x) = O(x^{\sigma_0+\kappa+\varepsilon})$ as $x \to \infty$,

(v) $\forall\ \varepsilon > 0,\ G_{\kappa+1}(x) = O(x^{\sigma_0+1+\kappa+\varepsilon})$ as $x \to \infty$.

Proof:

We show that (i) $\Longleftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The proof that (i) $\Longleftrightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) is similar.

To show that (i) $\Rightarrow$ (ii) suppose that $RH(\sigma_0)$ is true and consider first the case $\kappa = -1$. The method in Titchmarch [1], pages 282-283, can be modified to argue that $\zeta(s) = O(t^\varepsilon)$, $\frac{1}{\zeta(s)} = O(t^\varepsilon)$ as $t \to \infty$, for every $\sigma > \sigma_0$, and every $\varepsilon > 0$. Now let

$$f(s) = \zeta(2s)/\zeta(s).$$

Then for every $\sigma > \sigma_0$ and any $\varepsilon > 0$, $f(s) = O(t^\varepsilon)$ as $t \to \infty$, and by Titchmarch [1], page 6,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = f(s) \quad \text{for} \quad \sigma > 1.$$

Also it is clear that $f(1) = 0$. Using a procedure similar to that in Titchmarch [1], page 315 we thus get

$$S_{-1}(x) = \sum_{n\leq x} \frac{\lambda(n)}{n}$$

$$= \frac{1}{2\pi i}\int_{2-iT}^{2+iT} f(w+1)\,\frac{x^w}{w}\,dw + O(\frac{x^2}{T})$$

$$= \frac{1}{2\pi i}\int_{2-iT}^{\sigma_0-1+\delta-iT} + \int_{\sigma_0-1+\delta-iT}^{\sigma_0-1+\delta+iT} + \int_{\sigma_0-1+\delta+iT}^{2+iT} f(w+1)\frac{x^w}{w}\,dw \; +$$

$$+ \; 0\left(\frac{x^2}{T}\right) \, ,$$

$$= 0(T^{-1+\varepsilon}\, x^2) + 0(T^{\varepsilon}\, x^{\sigma_0-1+\delta})$$

as $x \to \infty$, provided $\varepsilon > 0$, and $0 < \delta < 1 - \sigma_0$. Hence, choosing $T = x^3$, for every $\varepsilon > 0$,

$$S_{-1}(x) = 0(x^{\sigma_0-1+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

i.e. (i) $\Rightarrow$ (ii) when $\kappa = -1$.

That (i) $\Rightarrow$ (ii) when $\kappa > -\sigma_0$ can now be deduced as follows. If $\kappa > -\sigma_0$ and $\varepsilon > 0$, then

$$S_\kappa(x) = \sum_{n\le x} (S_{-1}(n) - S_{-1}(n-1))n^{\kappa+1}$$

$$= \sum_{n\le x} S_{-1}(n)(n^{\kappa+1} - (n+1)^{\kappa+1}) \; +$$

$$+ \; S_{-1}(x)[x+1]^{\kappa+1}$$

$$= 0\left(\sum_{n\le x} n^{\kappa+\sigma_0-1+\varepsilon}\right) + 0(x^{\kappa+\sigma_0+\varepsilon})$$

$$= 0(x^{\kappa+\sigma_0+\varepsilon})$$

as $x \to \infty$, for every $\varepsilon > 0$.

To show that (ii) $\Rightarrow$ (i) suppose that for every $\varepsilon > 0$,

$$S_\kappa(x) = 0(x^{\kappa+\sigma_0+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Then, by partial summation,

$\sum_{n=1}^{\infty} \frac{\lambda(n)n^{\kappa}}{n^{s}}$ converges and represents an analytic function for

$\sigma > \sigma_0 + \kappa$.

Then from

$$\sum_{n=1}^{\infty} \frac{\lambda(n)n^{\kappa}}{n^{s}} = \frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)},$$

we see that $\zeta(s)$ is non-zero for $\sigma > \sigma_0$.

To show that (ii) $\Rightarrow$ (iii) suppose that $\kappa = -1$ or $\kappa > -\sigma_0$, and that

$$\forall \ \varepsilon > 0, \ S_{\kappa}(x) = O(x^{\sigma_0+\kappa+\varepsilon}) \ \text{ as } \ x \to \infty.$$

Then, via (i), also

$$\forall \ \varepsilon > 0, \ S_{\kappa+1}(x) = O(x^{\sigma_0+\kappa+1+\varepsilon}) \ \text{ as } \ x \to \infty.$$

But

$$S_{\kappa+1}(x) = \sum_{n \leq x} (S_{\kappa}(n) - S_{\kappa}(n-1))n$$

$$= -\sum_{n \leq x} S_{\kappa}(n) + S_{\kappa}(x)[x+1],$$

so that

(1)

$$H_{\kappa+1}(x) = \sum_{n \leq x} h_{\kappa+1}(n)$$

$$= \sum_{n \leq x} S_{\kappa}(n)$$

$$= [x+1]\, S_{\kappa}(x) - S_{\kappa+1}(x)$$

$$= O(x^{\sigma_0+\kappa+1+\varepsilon})$$

as $x \to \infty$, for every $\varepsilon > 0$.

To show that (iii) $\Rightarrow$ (i), note first that the estimate,

$$S_\kappa(x) = O(x^{\kappa+1}) \quad \text{as} \quad x \to \infty,$$

is trivial for $\kappa > -1$, and follows for $\kappa = -1$ from

$$S_{-1}(x) = \frac{1}{x} \sum_{n \leqslant x} \lambda(n)\left[\frac{x}{n}\right] + \frac{1}{x} \sum_{n \leqslant x} \lambda(n)\left\{\frac{x}{n}\right\}$$

$$= \frac{1}{x}\,[\sqrt{x}] + O(1)$$

as $x \to \infty$.

Consequently, using proposition 1, and Titchmarch [1], page 6, we have

$$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \sum_{n=1}^{\infty} \frac{\lambda(n)n^\kappa}{n^s}$$

(2) $$= s\int_1^\infty \frac{S_\kappa(x)}{x^{s+1}}\,dx$$

(3) $$= s\int_1^\infty \frac{x\,S_\kappa(x)}{x^{s+2}}\,dx$$

for $\sigma > \kappa + 1$, $\kappa \geqslant -1$.

Also, replacing s by $s + 1$, and κ by $\kappa + 1$ in (2), for $\sigma > \kappa + 1$, $\kappa \geqslant -2$

(4) $$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = (s+1)\int_1^\infty \frac{S_{\kappa+1}(x)}{x^{s+2}}\,dx \ .$$

Hence from (3) and (4), for $\sigma > \kappa + 1$, $\kappa \geqslant -1$,

(5) $$\frac{1}{s(s+1)}\,\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \int_1^\infty \frac{x\,S_\kappa(x) - S_{\kappa+1}(x)}{x^{s+2}}\,dx \ .$$

From (1) we easily see

$$H_{\kappa+1}(x) = x\,S_\kappa(x) - S_{\kappa+1}(x) +$$

$$+ O(x^{\kappa+1})$$

as $x \to \infty$, and so from (5) for $\sigma > \kappa + 1$, $\kappa \geqslant -1$,

(6) $$\frac{1}{s(s+1)} \frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \int_1^\infty \frac{H_{\kappa+1}(x)}{x^{s+2}} dx + E_\kappa(s),$$

where $E_\kappa(s)$ is analytic for $\sigma > \kappa$.

Finally if (iii) holds, i.e. if

$$\forall \ \varepsilon > 0, \ H_{\kappa+1}(x) = O(x^{\sigma_0+\kappa+1+\varepsilon}), \quad \text{as} \quad x \to \infty,$$

then the RHS of (6) is analytic for $\sigma > \sigma_0 + \kappa + \varepsilon$, and hence $\zeta(s)$ must be non-zero for $\sigma > \sigma_0$.

<u>Corollary</u>:

Let $\zeta(s)$ have zeros on $\sigma = \sigma_1 > 0$.

Let either $\kappa = -1$ or $\kappa > -\sigma_1$.

Then

(i) $\forall \ \varepsilon > 0, \ H_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon})$ as $x \to \infty$,

(ii) $\forall \ \varepsilon > 0, \ G_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon})$ as $x \to \infty$,

(iii) $\forall \ \varepsilon > 0, \ S_\kappa(x) = \Omega(x^{\kappa+\sigma_1-\varepsilon})$ as $x \to \infty$,

(iv) $\forall \ \varepsilon > 0, \ M_\kappa(x) = \Omega(x^{\kappa+\sigma_1-\varepsilon})$ as $x \to \infty$.

<u>Proof of (i)</u>:

Suppose the statement

$$\forall \ \varepsilon > 0, \ H_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon}) \quad \text{as} \quad x \to \infty,$$

is false. Then there exists $\varepsilon^* > 0$ such that

$$H_{\kappa+1}(x) = O(x^{\kappa+1+\sigma_1-\varepsilon^*}) \quad \text{as} \quad x \to \infty,$$

and hence from the previous proposition

$$\zeta(s) \text{ is zero free for } \sigma > \sigma_1 - \varepsilon^* ,$$

which contradicts the initial assumption. (ii), (iii), and (iv) follow similarly.

Note 1. Since $\zeta(s)$ does have zeros on $\sigma = \frac{1}{2}$ the statements of the corollary, with σ_1 replaced by $\frac{1}{2}$, are all true.

Note 2. The most familiar functions appearing in the literature are

$$S(x) = S_0(x) = \sum_{n\leq x} \lambda(n), \quad M(x) = M_0(x) = \sum_{n\leq x} \mu(n),$$

$$h(x) = h_0(x) = \sum_{n\leq x} \frac{\lambda(n)}{n}, \quad g(x) = g_0(x) = \sum_{n\leq x} \frac{\mu(n)}{n},$$

$$H(x) = H_0(x) = \sum_{n\leq x} h_0(n), \quad G(x) = G_0(x) = \sum_{n\leq x} g_0(n).$$

We now prove an extension of the previous proposition in a specialised case.

$$\text{Let } S^*(x) = \sum_{n\leq x} \lambda(n)\{\frac{x}{n}\}.$$

Proposition 3.

Let $1 > \sigma_0 \geq \frac{1}{2}$. The following statements are equivalent:

(i) $\forall\ \varepsilon > 0,\ H(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(ii) $\forall\ \varepsilon > 0,\ h(x) = O(x^{\sigma_0-1+\varepsilon})$ as $x \to \infty$,

(iii) $\forall\ \varepsilon > 0,\ S(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(iv) $\forall\ \varepsilon > 0,\ S^*(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(v) $\forall\ \varepsilon > 0,\ S(x) - S^*(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(vi) $RH(\sigma_0)$.

Proof:

We have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi) from proposition 2.

From (1),

$$xh(x) = H(x) + S(x) + O(1) \quad \text{as} \quad x \to \infty.$$

Also,

$$(7) \qquad xh(x) - S^*(x) = \sum_{n \leq x} \lambda(n) \left[\frac{x}{n}\right]$$

$$= [\sqrt{x}] \ ,$$

and hence from these two equations

$$(8) \qquad H(x) = S^*(x) - S(x) + O(x^{\frac{1}{2}}) \quad \text{as} \quad x \to \infty.$$

From (7), (ii) $\Leftrightarrow$ (iv), and from (8), (i) $\Leftrightarrow$ (v), thus completing the proof.

Note 3. In the previous proposition (ii) $\Rightarrow$ (i) holds for every pair of functions k, K such that

$$K(x) = \sum_{n \leq x} k(n), \text{ and in this}$$

sense (i) is weaker than (ii), and in the next section we develop this theme further.

Note 4. A corresponding result to proposition 3 holds for the functions

$$G(x),\ g(x),\ M(x),\ M^*(x).$$

Note 5. Turan's conjecture that $h(x) > 0$, for $x > 1$, has been upset by numerical investigation (Haselgrove, C.B. [1]) but we note in the next section that the argument of Lehmer and Selberg [1], that $G(x)$ changes sign infinitely often as $x \to \infty$, does not apply to $H(x)$ if RH($\frac{1}{2}$) is true.

Section 2.

Further statements equivalent to $RH(\sigma_0)$.

The notion of 'weakness' we mention in note (3), section 1, manifests itself in higher averages.

Let $$A_{-1}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n},$$

and for any integer $k \geq 0$ let

$$A_k(x) = \sum_{n \leq x} A_{k-1}(n).$$

In this notation,

$$h(x) = A_{-1}(x),$$

and

$$H(x) = A_0(x).$$

In this section we prove:

Proposition 1.

For any fixed integer $r \geq -1$, the following statements are equivalent:

(i) $RH(\sigma_0)$,

(ii) For every $\varepsilon > 0$, $A_r(x) = O(x^{\sigma_0 + r + \varepsilon})$ as $x \to \infty$.

Before proceeding to the proof we establish some helpful lemmas:

Lemma 1.

For every integer $r \geq -1$,

$$A_r(x) = \frac{1}{(r+1)!} x^{r+1} \sum_{n \leq x} \frac{\lambda(n)}{n} \left(1 - \frac{n}{x}\right)^{r+1} + O(x^r) \quad \text{as } x \to \infty.$$

Proof:

For $r = -1$, the truth of the above statement is seen from the definition of $A_{-1}(x)$.

Also,

$$A_0(x) = \sum_{k \le x} A_{-1}(k)$$

$$= \sum_{k \le x} \sum_{n \le k} \frac{\lambda(n)}{n}$$

$$= \sum_{n \le x} \frac{\lambda(n)}{n} \sum_{n \le k \le [x]} 1$$

$$= \sum_{n \le x} \frac{\lambda(n)}{n} ([x] - n + 1)$$

$$= \sum_{n \le x} \frac{\lambda(n)}{n} (x-n) + O(1)$$

$$= \frac{1}{1!} x^1 \sum_{n \le x} \frac{\lambda(n)}{n} (1-\frac{n}{x})^1 + O(1), \text{ as } x \to \infty,$$

and we see the proposition is true for $r = 0$.

Now suppose the proposition is true for $r = R \geq 0$.

Then

$$A_{R+1}(x) = \sum_{k \le x} A_R(k)$$

$$= \sum_{k \le x} \frac{1}{(R+1)!} k^{R+1} \sum_{n \le k} \frac{\lambda(n)}{n} (1-\frac{n}{k})^{R+1} + $$

$$+ O\left(\sum_{k \le x} k^R\right)$$

$$= \frac{1}{(R+1)!} \sum_{k \le x} \sum_{n \le k} \frac{\lambda(n)}{n} (k-n)^{R+1} + O(x^{R+1})$$

$$= \frac{1}{(R+1)!} \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{n \leq k \leq [x]} (k-n)^{R+1} + O(x^{R+1})$$

(1) $$= \frac{1}{(R+1)!} \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{0 \leq k \leq [x]-n} k^{R+1} + O(x^{R+1}) \quad \text{as} \quad x \to \infty.$$

Now

(2) $$\sum_{k=1}^{b} k^{R+1} = \frac{1}{R+2} b^{R+2} + \sum_{i=1}^{R+1} C_{R+1,i}\, b^{i}$$

where the coefficients $C_{R+1,i}$ are independent of b. Consequently, from (1) and (2),

(3) $$A_{R+1}(x) = \frac{1}{(R+2)!} \sum_{n \leq x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} +$$

$$+ \frac{1}{(R+1)!} \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} ([x]-n)^{i} +$$

$$+ O(x^{R+1}).$$

But

$$\sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} ([x]-n)^{i}$$

$$= \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} \sum_{t=0}^{i} \binom{i}{t} [x]^{i-t} (-n)^{t}$$

$$= \sum_{i=1}^{R+1} C_{R+1,i} \sum_{t=0}^{i} \binom{i}{t} [x]^{i-t} (-1)^{t} \sum_{n \leq x} \lambda(n) n^{t-1}$$

$$= O\left(\sum_{i=1}^{R+1} \sum_{t=0}^{i} x^{i}\right)$$

$$= O(x^{R+1})$$

as $x \to \infty$, since, as noted in section 1, $\sum_{n \leq x} \lambda(n) n^{t-1} = O(x^{t})$, as $x \to \infty$, for each integer $t \geq 0$.

Thus it follows from (3) that

$$A_{R+1}(x) = \frac{1}{(R+2)!} \sum_{n\le x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} + O(x^{R+1}) \tag{4}$$

as $x \to \infty$.

Finally,

$$\sum_{n\le x} \frac{\lambda(n)}{n} ([x]-n)^{R+2}$$

$$= \sum_{n\le x} \frac{\lambda(n)}{n} ((x-n)-\{x\})^{R+2}$$

$$= \sum_{n\le x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{1\le t\le R+2} (x-n)^{R+2-t}(-1)^t\{x\}^t \binom{R+2}{t}$$

$$= \sum_{n\le x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{1\le t\le R+2} \binom{R+2}{t} \sum_{0\le s\le R+2-t} x^{R+2-t-s}(-1)^s n^s (-1)^t \{x\}^t \binom{R+2-t}{s}$$

$$= \sum_{n\le x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{1\le t\le R+2} \binom{R+2}{t} \sum_{0\le s\le R+2-t} (-1)^{s+t} \{x\}^t x^{R+2-t-s} \binom{R+2-t}{s} \sum_{n\le x} \lambda(n) n^{s-1}$$

But, as noted in section 1, for $s \ge 0$

$$\sum_{n\le x} \lambda(n) n^{s-1} = O(x^s)$$

as $x \to \infty$. Hence

$$\sum_{n\le x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} - \sum_{n\le x} \frac{\lambda(n)}{n} (x-n)^{R+2} \tag{5}$$

$$= O\left\{ \sum_{1\le t\le R+2} \sum_{0\le s\le R+2-t} x^{R+2-t} \right\} = O(x^{R+1})$$

as $x \to \infty$. The lemma now follows from (4) and (5), and the principle of induction.

Recalling the notation

$$S_\kappa(x) = \sum_{n \leq x} \lambda(n) n^\kappa$$

we next have

<u>Lemma 2.</u>

For every integer $r \geq -1$,

$$A_r(x) = \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa x^{r+1-\kappa} S_{\kappa-1}(x) +$$

$$+ O(x^r) \quad \text{as} \quad x \to \infty.$$

<u>Proof:</u>

From lemma (1)

$$A_r(x) = \frac{1}{(r+1)!} \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} x^{r+1-\kappa} (-1)^\kappa n^\kappa +$$

$$+ O(x^r)$$

$$= \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} x^{r+1-\kappa} (-1)^\kappa \sum_{n \leq x} \lambda(n) n^{\kappa-1} +$$

$$+ O(x^r)$$

as $x \to \infty$.

Thus

(1)
$$A_r(x) = \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa x^{r+1-\kappa} S_{\kappa-1}(x) +$$

$$+ O(x^r) \quad \text{as} \quad x \to \infty.$$

Lemma 3.

For $\sigma > r+1$, and every integer $r \geq -1$,

$$\int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx = \frac{1}{s(s-1)\ldots(s-r-1)}\ \frac{\zeta(2s-2r)}{\zeta(s-r)} + $$

$$+ P_r(s)$$

where $P_r(s)$ is analytic for $\sigma > r$.

Proof:

We have noted in (2), Section 1, that

$$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = s\int_1^\infty \frac{S_\kappa(x)}{x^{s+1}}\,dx$$

for $\kappa \geq -1$, and $\sigma > \kappa+1$.

Writing $s-r+\kappa$ for s in this formula we have

$$\frac{\zeta(2s-2r)}{\zeta(s-r)} = (s-r+\kappa)\int_1^\infty \frac{S_\kappa(x)}{x^{s-r+\kappa+1}}\,dx$$

for $\sigma > r+1$, with $\kappa \geq -1$.

Hence

(7) $$\int_1^\infty \frac{x^{r-\kappa+1}S_{\kappa-1}(x)}{x^{s+1}}\,dx = \frac{1}{(s-r+\kappa-1)}\ \frac{\zeta(2s-2r)}{\zeta(s-r)}$$

for $\sigma > r+1$ with $\kappa \geq 0$.

Consequently, from lemma 2,

$$\int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx$$

$$= \frac{1}{(r+1)!}\sum_{\kappa=0}^{r+1}\binom{r+1}{\kappa}(-1)^\kappa \int_1^\infty \frac{x^{r+1-\kappa}\ S_{\kappa-1}(x)\,dx}{x^{s+1}} + P_r(s),$$

where $P_r(s)$ is analytic for $\sigma > r$.

Then from (7) we have for $\sigma > r + 1$,

(8) $$\int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx$$

$$= \frac{1}{(r+1)!}\sum_{\kappa=0}^{r+1}\binom{r+1}{\kappa}(-1)^\kappa \frac{1}{(s-r+\kappa-1)}\ \frac{\zeta(2s-2r)}{\zeta(s-r)}$$

$$+ P_r(s)\,.$$

Using the 'cover up' rule for partial fractions we easily see that

$$\frac{1}{(r+1)!}\sum_{\kappa=0}^{r+1}\binom{r+1}{\kappa}(-1)^\kappa \frac{1}{(s-r+\kappa-1)}$$

$$= \frac{1}{s(s-1)\ldots(s-r-1)}\,,$$

and hence from (8),

(9) $$\int_1^\infty \frac{A_r(x)}{x^{s+1}} = \frac{1}{s(s-1)\ldots(s-r-1)}\ \frac{\zeta(2s-2r)}{\zeta(s-r)} +$$

$$+ P_r(s)$$

for $\sigma > r + 1$, where $P_r(s)$ is analytic for $\sigma > r$.

<u>Proof of proposition 1:</u>

For integer $r \geq -1$ let T_r be the statement:

For every $\varepsilon > 0$, $A_r(x) = O(x^{\sigma_0+r+\varepsilon})$ as $x \to \infty$.

From proposition 3, section 1, we have

$$\mathrm{RH}(\sigma_0) \iff T_{-1} \iff T_0\,.$$

Clearly, $T_r \Rightarrow T_{r+1}$ for all $r \geq 0$. It thus suffices to show

$T_r \Rightarrow \mathrm{RH}(\sigma_0)$ for any fixed $r \geqslant -1$, and this follows readily from (9).

Note 1. With $B_{-1}(x) = \sum_{n \leqslant x} \frac{\mu(n)}{n}$

and

$$B_k(x) = \sum_{n \leqslant x} B_{k-1}(n)$$

for integer $k \geqslant 0$, the method of proof of the preceding proposition leads to an analogue of lemma (3). Namely,

for $\sigma > r + 1$, and integer $r \geqslant -1$,

$$\int_1^\infty \frac{B_r(x)}{x^{s+1}}\, dx = \frac{1}{s(s-1)\ldots(s-r-1)\zeta(s-r)} + Q_r(s), \tag{10}$$

where $Q_r(s)$ is analytic for $\sigma > r$. We consequently have

Proposition 2.

For any fixed integer $r \geqslant -1$, the following statements are equivalent

(i) $\mathrm{RH}(\sigma_0)$,

(ii) For every $\varepsilon > 0$, $B_r(x) = O(x^{\sigma_0 + r + \varepsilon})$ as $x \to \infty$.

Proof: c.f. Proposition 1.

Note 2. Although we are concentrating mainly on the Möbius function and the Liouville function the preceding propositions apply to the class of functions $\{\tau^{(k)}\}$ defined for $k = 2, 3, \ldots$ by

$$\sum_{n=1}^\infty \frac{\tau^{(k)}(n)}{n^s}\zeta(s) = \zeta(ks), \quad (\sigma > 1),$$

Here we have $\tau^{(2)} \equiv \lambda$, and, in a sense, $\tau^{(\infty)} \equiv \mu$.

Section 3.

Some results on the oscillatory behaviour of certain summatory functions involving μ and λ.

Let $A_r(x)$ and $B_r(x)$ be defined as in section 2. Let $\bar{\sigma}$ satisfy $\frac{1}{2} \leqslant \bar{\sigma} < 1$ and be such that $\zeta(s) = 0$ has a solution with $\sigma \geqslant \bar{\sigma}$. From propositions 1 and 2, section 2, it follows that

$$\forall \quad \varepsilon > 0, \quad A_r(x) = \Omega(x^{r+\bar{\sigma}-\varepsilon}),$$

and

$$\forall \quad \varepsilon > 0, \quad B_r(x) = \Omega(x^{r+\bar{\sigma}-\varepsilon}),$$

as $x \to \infty$. Actually, we can say more than this.

Proposition 1.

Let r be an integer, $r \geqslant -1$, and let K be any real number. Then for every $\varepsilon > 0$,

$$B_r(x) - K\, x^{r+\bar{\sigma}-\varepsilon}$$

changes sign infinitely often as $x \to \infty$.

i.e. $$\forall \quad \varepsilon > 0, \quad B_r(x) = \Omega_{\pm}(x^{r+\bar{\sigma}-\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Proof:

For $\sigma > r + 1$, $r \geqslant 0$, let the Dirichlet series $L_r(s)$ be defined by

$$L_r(s) = \sum_{n=1}^{\infty} \frac{B_{r-1}(n) - Kn^{r-1+\bar{\sigma}-\varepsilon}}{n^s}, \quad \text{where} \quad 0 < \varepsilon < \bar{\sigma}.$$

From proposition 1, section 1,

$$L_r(s) = \sum_{n=1}^{\infty} \frac{B_{r-1}(n)}{n^s} - K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$$

$$= s\int_1^{\infty} \frac{B_r(x)}{x^{s+1}}\,dx - K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$$

Hence, from 10, section 2,

$$(1) \qquad L_r(s) = \frac{s}{s(s-1)\;\ldots.(s-r-1)\,\zeta(s-r)} - K\zeta(s-r+1-\bar{\sigma}+\varepsilon) + $$

$$+ Q_r(s),$$

where $Q_r(s)$ is regular for $\sigma > r$.

Suppose that the coefficients of the series for $L_r(s)$ are eventually of one sign. Then by a classical theorem of Landau, the series has a singularity at the real point on the line of convergence of the series. But the first term in (1), $\frac{1}{(s-1)(s-2)\ldots(s-r-1)\zeta(s-r)}$, has singularities at $s = 1, 2, \ldots, r$ and $s = r + \rho$, where ρ is a zero of $\zeta(s)$. Since $\zeta(s)$ has no real zeros with $s \geq 0$ the first term has no real singularities with $\sigma > r$. The second term in (1), $K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$, has no singularities at all for $\sigma > r + \bar{\sigma} - \varepsilon$. Hence $L_r(s)$ has no real singularity for $\sigma > r + \bar{\sigma} - \varepsilon$, and the abscissa of convergence of the Dirichlet series for $L_r(s)$ must be less than or equal to $r + \bar{\sigma} - \varepsilon$. Hence $L_r(s)$ is analytic for $\sigma > r + \bar{\sigma} - \varepsilon$, and thus, from (1), $\zeta(s)$ must be non-zero for $\sigma > \bar{\sigma} - \varepsilon$, which contradicts the definition of $\bar{\sigma}$. If follows that the coefficients of the Dirichlet series for $L_r(s)$ cannot be ultimately of one sign, and this completes the proof.

Corollary 1:

Let r be an integer, $r \geq -1$, and let K be any real number.

Then $\forall \ \varepsilon > 0, \ B_r(x) - K\,x^{r+\frac{1}{2}-\varepsilon}$

changes sign infinitely often as $x \to \infty$.

<u>Proof:</u>

This follows since $\bar{\sigma} \geq \frac{1}{2}$.

As a corollary to the method of proof of proposition 1 we also have

<u>Corollary 2.</u>

Let r be an integer, $r \geq -1$, and let K be any real number. Let $1 \geq \sigma_0 \geq \frac{1}{2}$. If $B_r(x) - K\,x^{r+\sigma_0}$ is eventually of one sign as $x \to \infty$, then $\mathrm{RH}(\sigma_0)$ is true.

<u>Proof:</u>

Let $B_r(x) - K\,x^{r+\sigma_0}$ be eventually of one sign as $x \to \infty$. Then with σ_0 playing the role of $\bar{\sigma}$ in the equations leading up to (1) we find

$$L_r(s) = \frac{1}{(s-1)\ldots.(s-r-1)\,\zeta(s-r)} - K\,\zeta(s-r+1-\sigma_0) +$$

$$+\ Q_r(s),$$

where $Q_r(s)$ is regular for $\sigma > r$. As in proposition 1 we then have $L_r(s)$ analytic for $\sigma > \sigma_0 + r$ and consequently $\zeta(s) \neq 0$ for $\sigma > \sigma_0$.

[illegible] Analagous results to proposition 1 hold for the corresponding summatory functions associated with $\tau^{(k)}$ for $k = 3, 4, \ldots$, where we recall

$$\sum_{n=1}^{\infty} \frac{\tau^{(k)}(n)}{n^s}\,\zeta(s) = \zeta(ks), \quad (\sigma > 1).$$

However, for $k = 2$, $\tau^{(2)} \equiv \lambda$, and the equation corresponding to (10) is

$$L_r(s) = \frac{1}{(s-1)(s-2)\dots(s-r-1)} \frac{\zeta(2s-2r)}{\zeta(s-r)} +$$

$$- K\zeta(s-r+1-\bar{\sigma}+\varepsilon) + P_r(s).$$

The pole of $\zeta(2s-2r)$ at $s = r + \frac{1}{2}$ prevents the argument in proposition 1 following here in the case $\bar{\sigma} = \frac{1}{2}$. But for $\bar{\sigma} > \frac{1}{2}$ the corresponding result holds.

i.e.

<u>Proposition 2.</u>

Let $\bar{\sigma}$ satisfy $\frac{1}{2} < \bar{\sigma} < 1$ and be such that $\zeta(s) = 0$ has a solution with $\sigma \geq \bar{\sigma}$. Let r be an integer, $r \geq -1$. Then for every $\varepsilon > 0$,

$$A_r(x) = \Omega_{\pm}(x^{r+\bar{\sigma}-\varepsilon})$$

as $x \to \infty$.

<u>Proof:</u>

Similar to that of proposition 1. A result corresponding to corollary 1 cannot be stated here for the $A_r(x)$, since proposition 2 assumes $\bar{\sigma} > \frac{1}{2}$, and if RH($\frac{1}{2}$) is true it is conceivable that the $A_r(x)$ are eventually of one sign as $x \to \infty$ for some $r \geq R > -1$.

However, we do have an analogue of corollary 2, for the $A_r(x)$. Namely,

<u>Corollary 3.</u>

Let r be an integer, $r \geq -1$, and let K be any real number. [illegible] $1 \geq \sigma_0 \geq \frac{1}{2}$. If $A_r(x) - Kx^{r+\sigma_0}$ is eventually of one sign as [illegible] then RH(σ_0) is true.

Proof:

Similar to that of corollary 2.

Note 2. These results improve and generalise the result of Lehmer and Selberg [1], that $B_0(x) - K$ changes sign infinitely often as $x \to \infty$, and generalise the well known result that if $\frac{H(x)}{x^{\frac{1}{2}}}$ is either bounded above or below then RH(½) is true.

References

Backlund, R.
[1] Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion, Öfversigt Finska Vetensk. Soc. 61, (1918-1919), No. 9.

Berlowitz, B.
[1] Extensions of a theorem of Hardy, Acta. Arith. 14, (1967-1968), p. 203-207.

Berndt, B.C.
[1] On the zeros of Riemann's zeta-function, Proc. Amer. Math. Soc. 22, (1969), p. 183-188.

Beuring, A.
[1] Analyse de la loi asymptotique de la distribution des nombres premiers generalises, Acta. Math. 68, (1937), p. 255-291.

Braun, P.B.
[1] See A. Zulauf [1].

[2] Unpublished notes.

[3] (with A. Zulauf) A problem connected with the zeros of Riemann's zeta-function, Proc. Amer. Math. Soc. 36, (1972). No. 1, p. 18-20.

[4] A series representation for Riemann's Ξ-function, Mathematics Research Report, Univ. Of Waikato, No. 7, (1972).

[5] (with A. Zulauf) An elementary connection between the orders of $M(x)$ and $\psi(x) - x$, Mathematics Research Report, Univ. of Waikato, No. 58, (1978).

[6] See A. Zulauf [6].

Davenport, H.
[1] Multiplicative Number Theory, Markham Publishing Co. 1967.

Edwards, H.M.
[1] Riemann's Zeta Functions, Academic Press, 1974.

Formenko, O.M.
[1] Two hypotheses in the theory of prime numbers, (Russian), Rev. Math. Pures Appl. 6, (1961), p. 745-746.

Frankel, J., and E. Landau.
[1] Les suites de Farey et le problème des nombres premiers, Göttinger Nachrichten. (1924), p. 198-206.

Gelfond, A.O., and Yu. V. Linnik.
[1] Elementary Methods in Analytic Number Theory. Rand McNally and Co., (1965).

Haselgrove, C.B.
[1] A disproof of a conjecture of Polya, Mathematika, 5, (1958), p. 141-145.

[2] Tables of the Riemann zeta function, Royal Soc. Math. Tables, Vol. 6. (1960), U.P. Cambridge.

Huxley, M.N.
[1] On the difference between consecutive primes, Inventiones math. 15, (1972), p. 164-170.

Ingham, A.E.
[1] Reviews in Number Theory, Amer. Math. Soc. Vol. 4, Review article N. 44-16, p. 276.

Jahnke, Emde, and Losch,
[1] Tables of Higher Functions, ((vi) edition), McGraw-Hill, (1960).

Jutila, M.
[1] On Linnik's constant, Math. Scad., 41, (1977), p. 45-62.

Kopriva, J.
[1] O jedom vztahu Fareyovy rady k Riemannove domnence o nulovych bodech funkce ξ, Časopis pro pestovani matematiky, 78, (1953), p. 49-55.

Lehmer, D.H., and S. Selberg.
[1] A sum involving the function of Möbius, Acta. Arith. VI (1960), p. 111-114.

Le Veque, W.J.
[1] Reviews in Number Theory, Amer. Math. Soc., Vol. 4, (1974), p. 335-367.

Levinson, N.
[1] On Theorems of Berlowitz and Berndt, Jour. No. Theory 3, (1971), p. 502-504.

Littlewood, J.E.
[1] Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $R(s) > ½$, C.R. 154 (1912), p. 263-266.

Montgomery, H.L.
[1] Review 3338, Math. Reviews, 45, (1973), p. 613.

Selberg, S.
[1] Über eine Vermutung von P. Turan, Norske Vid. Selsk. Forh., Trondheim 29, (1956), p. 33-35.

Spira, R.
[1] The integral representation for the Riemann Ξ-function, Jour. No. Theory, 3, No. 4, (1971), p. 498-501.

Titchmarch, E.C.

[1] The Theory of the Riemann Zeta-function, Oxford 1951.

[2] The Theory of Functions, O.U.P. (1939).

Zulauf, A.

[1] (with P.B. Braun) General Theorems on Special Divisor Problems, Mathematics Research Report, Univ. of Waikato, No. 57, (1978).

[2] Unpublished notes.

[3] See P.B. Braun [3].

[4] The Distribution of Farey Numbers, Journal für die reine und angewandte Mathematik, Band 289, p. 209-213.

[5] See P.B. Braun [5].

[6] (with P.B. Braun) Generalized Integers and Generalized Logarithms, Mathematics Research Report, Univ. of Waikato, No. 59 (1978).