The argument form used in the twin prime problem – Peter Braun

The inductive natural numbers provide the theoretical model for our counting.

It is inbuilt that they cannot be counted since the natural numbers stay ahead of the person or machine counting them using the notion of successor.

The notion of unboundedness is closely linked to the successor idea.

The tortoise and hare argument is valid here– the axioms of Peano ensure that the tortoise stays ahead. The axioms do not insist that all natural numbers are imaginable as counting numbers but only that the notion of successor in the logical implication process leads to 'complete' proof.

The theoretical model then extends the notion of counting.

In the sentence – 'Let n be a natural number …' n is pictured as a counting number,

and this is a practical orientation to assist the mathematical activity going on.

Consider the notion of theoretical counting:

The grains of sand on a beach are finite and it is possible to imagine them being counted. Without actually counting them we are comfortable with the idea that the notion is sound. This may be called a theoretical count.

However, even allowing for theoretical counting, we cannot imagine counting all the natural numbers because of the ever ready supply of further numbers.

The alignment between the counting numbers and the natural numbers is thus capable of further extension.

The existence of a natural number τ which is not a counting number is a theoretical construct used to increase the alignment between counting numbers and the formal natural numbers.

Boundaries

The above area of thinking may be thought of as being about boundaries.

Generally a boundary will create a distinction between A and B.

If A and B are both defined independently the boundary is likely to be more tangible in the sense that a listing of the defining characteristics for A and B are given in a positive sense. B (for example) is not defined through 'what A is not'. In the latter case the boundary is less easy to understand.

For example with a simple closed curve – if we take off from a point on the inside in a straight line we intersect the curve. On the outside there is a choice of directions for which continuous straight line segment travel which will avoid intersecting the curve.

The regions I (inside) and O (outside) possibly have independent defined meanings but the notion of unbounded needs to be hauled in and defined discretely in order to be imagined and understood. We may think we can imagine 'unbounded' but what we mean inevitably reduces to something quite distinctly bounded.

The words boundary and bounded look alike for good reason. Mathematics attempts to use technical terms which make sense.

With the natural numbers we want to introduce a natural number τ which is not a counting number. We may need a boundary so that a sensible distinction may be made between counting and induction. Alternatively, it would be sufficient to establish that no inconsistency would ever arise in the assumption.

In the former the 'boundary' is defined by the counting numbers or what we have talked about above as the theoretical counting numbers. Any realm of imagination in which a theoretical count is contemplated will be assumed valid as it relates to counting experience. The boundary is the limit of imagination. Beyond the boundary – what is there? Or more pertinently what do we want to make up?

We need some continuity so if we have an entity τ we would want it to be an inductive number. The continuity request here is under the essential law that if a distinction may be drawn between A and B at a boundary there must be some commonality between A and B.

With the Jorden curve theorem the commonality is seen in the boundary points.

Both A and B are 'adjacent' to boundary points and each is a collection of points.

With the natural numbers each counting number admits a technical definition which aligns with imagination and counting (and induction) but the inductive number $\tau$is

a number outside of this definition. It cannot be ruled out of existence because not all natural numbers can be imagined as counting numbers.

In other words no inconsistency will appear in assuming τ.

We note that the status of τ is not relevant to the truth or falsity of existing theorems. These theorems are established. They remain with their truth status.

τ cannot force some contradictory state of affairs simply BECAUSE it is assumed to be an inductive number. In a formal sense τ is just a label for a type of inductive number.

Thus 'rules' type activity and manipulation in number theory is not challenged.

The addition of τ will never interfere with the true/false status of a theorem obtained without τ. The inclusion of τ is thus independent of the other axioms.

Using τ in reasoning

But what happens if we wish to use τ in proving theorems or having it appear in strings of propositions which are somehow logically linked.

How does logic like τ?

Here – asking the question doesn't mean the question is altogether sensible.

We first need some mathematical activity in a reasoning realm which uses the τ construction.

We need one truth.

For each n let P(n) is a statement about the natural number n which exists as an object of thought and which needs to be examined numerically to understand the meaning of P(n) being true and the meaning of P(n) being false.

Then $P(\tau)$ is unprovable as a theorem in Peano arithmetic.

If Q(N) is a theorem which has been proven using the principle of mathematical induction we would take $Q(\tau)$ to be true as a logic extension to normal inductive reasoning.

The argument form we wish to establish is:-

P(1) true, $P(N+1) \rightarrow P(N)$ (the truth of P(N) follows from the truth of P(N+1)), then

P(N) is true for N= 1,2…. (the τ argument).

The argument for this is in the following paper on the generalisation of the twin prime problem.

We do however need to narrow the class of theorems for which this is valid as artificial counter examples can be constructed.

The idea here is that P(N) is a sensible, substantial statement involving N which is not artificially analytical like $f_v f_v f_v$ ....f_v (f repeated N times) or contains other tautologies or fallacies. We cannot sensibly contemplate $f_v f_v f_v$ ....f_v

(f repeated N times) since it is false by definition. We need P(N) to be an object of thought where without proof it is logically possible for P(N) to be either true, false or unprovable. We reject P(N) where the truth value of P(N) is determined by the statement of P(N) without proof (an analytical construction) .

Suppose for each N that

$P(1) \equiv P(2) \equiv \ldots\ldots\ldots \equiv P(N) \equiv \ldots\ldots\ldots \equiv P(\tau)$ ....(1)

in the usual sense of logical equivalence and $P(N+1) \rightarrow P(N)$.

Example1

$P(N) \equiv \sum n = (1/2)N(N+1) + 32$ (Summing the first N natural numbers)

P(1) is false. Such attempts at counter examples which tamper with the structure of an inductive theorem are unlikely to satisfy all the conditions. Indeed if P(N) has been proven by induction then we have $P(N+1) \rightarrow P(N)$ and P(1) is true.

If the 'reverse' induction is going to work we need P(1) as an anchor in much the same way as normal induction.

There is another type of proposition using ≤ which needs to be ruled out.

Example 2 (Preliminary comments)

We use Fermat's last theorem to illustrate the case but it is relatively easy to construct the same mechanism in other settings.

We call it an invalid form of the τ argument.

Recall we are considering theorems P(N) in number theory about the natural number N.

Here we assume that P(N) is a sensible object of thought which is either true, false or unprovable in Peano arithmetic and satisfies

$P(N) \equiv P(1)\wedge P(2)\wedge\ldots\wedge P(N)$.

We then have $P(N+1) \rightarrow P(N)$.

We are not concerned with algebraic manipulation in a formal logic system but with propositions P(N), S(N) ... where we are interested in whether P(N) follows from S(N) and so on. That is – the focus of interest is in the necessary/sufficient activity which goes on in mathematics to try and establish the truth of propositions.

The purpose is to try and locate explanation in general reasoning rather than outcome within a limited formal system.

Now consider the theorem

$T(N) \equiv x^n + y^n = z^n$ has a non-trivial solution in integers for $1 \leq n \leq N$.

If we let

$P(n) \equiv x^n + y^n = z^n$ has a non-trivial solution in integers

then clearly

$T(N) = P(1)\wedge P(2)\wedge\ldots\wedge P(N)$.

However, what would $T(N) \equiv P(N)$ imply?

If $T(N) \equiv P(N)$ in terms of mathematical activity then a non–trivial solution to $x^N + y^N = z^N$ would imply non trivial solutions to $x^n + y^n = z^n$ for $1 \leq n \leq N$ and whereas this is true for $N = 1$ and $N = 2$ there is no known proof (for example) that we are able to derive a solution $x^2 + y^2 = z^2$ given a non–trivial solution

$A^3 + B^3 = C^3$.

That is, we do not have a proof along the lines:-

Suppose $A^3 + B^3 = C^3$ is a non trivial solution.

Then …

And so ….

In then follows that …

……

……

Hence $x_0^2 + y_0^2 = z_0^2$.

No such argument appears to be accessible.

Problem

With P(N), T(N) as defined above:

prove that if P(N) is true then each of P(1), P(2)…P(N) is true.

This type of activity is legitimately used in reductio ad absurdum proof.

We assert that P(N-1) does not follow from P(N) and this is what we are assuming when T(N) is put forward as an example in the following discussion.

The state of play with Fermat's theorem is not of interest in this context and the known result need not be taken into account in considering the theorem.

The following then attempts to 'tighten' up somewhat on proposition sequences for which the τ argument is valid.

Definition:

A theorem T(N) which satisfies

$T(N) \equiv T(1)\wedge T(2)\wedge\ldots\wedge T(N)$ and $T(N) \rightarrow T(N\text{-}1)$

is called synthetic if with

$T(N) \equiv P(1)\wedge P(2)\wedge\ldots\wedge P(N)$

we have

$T(N) \equiv P(N)$

(by this we mean the truth of the theorem T(N) follows from the truth of P(N) and vice versa).

Then, under certain assumptions,

$T(N) \equiv x^n + y^n = z^n$ has a non-trivial solution in integers for $1 \leq n \leq N$

is an example of a non-synthetic proposition.

Inductive theorems.

Let S(N) be a theorem which has been proven true using mathematical induction.
Clearly $S(N) \equiv S(1)\wedge S(2)\wedge\ldots\wedge S(N)$.
Suppose $S(N) \equiv P(1)\wedge P(2)\wedge\ldots\wedge P(N)$ where $P(N) \rightarrow P(N\text{-}1)$.
Then $S(N) \equiv P(N)$.
Thus inductive theorems are synthetic.
Thus although $S(\tau)$ is unprovable it may be taken to be true in the sense that all numerical evidence will support this orientation.

The idea then is to pursue synthetic theorems in arithmetic using the argument
P(1) is true, $P(N+1) \rightarrow P(N)$, $P(\tau)$ is unprovable, P(N) may be taken to be true without contradiction.
The exclusion of non-synthetic propositions must be driven by reason rather than the desire to slip out of a counter example to the τ argument.
It seems to hinge on the primitiveness of the generalisation.
In the case of the twin prime generalisation P(N) has some sort of essential simplicity which is directly about a problem.
With the Fermat example T(N) is an aggregate of propositions P(1), P(2) …P(N) which are not in general related unit stepwise.
$T(\tau)$ is essentially the proposition that $x^N + y^N = z^N$ has a non-trivial solutions in integers for each natural number N.
Behind T(N) is the more elementary proposition P(N) and the crucial step
$P(N) \rightarrow P(N\text{-}1)$ is missing – believed impossible.
The use of the word synthetic is intended to imply that the failure of the τ argument for non-synthetic τ arguments is built in to the statement of the generalised argument – an inverted analogue of the analytical statement.
By looking at the ‘unbundled’ propositions we may be losing the ‘real’ inductive connection $P(N+1) \rightarrow P(N)$ necessary for the τ argument.

Example 3 (including the twin prime problem)

Let a(1), a(2) ...a(N) be any sequence of counting numbers with

$a(1) < a(2) < a(3) < \ldots < a(N)$

and the property that there is no full residue class modulo any prime number in

{a(1), a(2), …a(N)}.

Let

$P(N) \equiv$ There exist unbounded numbers of n such that

n+a(1), n+a(2), ..., n+a(N) are all prime numbers.

The assertion here is that P(N) is synthetic and that the τ argument has validity here.

The idea is that P(N) synthetic allows $P(\tau)$ to be taken as unprovable and used as an anchor in reverse induction in much the same way as obtaining P(1) to be true is the crucial anchor in normal induction.

Symbolically: $[\ P(1), P(N) \rightarrow P(N+1)] = [\ P(1), P(\tau), P(N+1) \rightarrow P(N)\ ]$

for synthetic theorems.

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