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Seven Steps to the Riemann Hypothesis (RH)
by
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Notation and usage

The appearance of ϵ in a proposition will mean for all $\epsilon > 0$ ……. .
For $O(x)$, $o(x)$, $\Omega_+(x)$, $\Omega_-(x)$ and $\Omega_{+-}(x)$, $x\to\infty$ will be assumed.

Thus: for all $\epsilon>0$, $A(x) = O(x^{(\Delta+\epsilon)})$ as $x\to\infty$, is simply written $A(x) = O(x^{(\Delta+\epsilon)})$.

Let γ be the lub of numbers θ such that $A(x) = O(x^\theta)$. Then we write $A(x) = |O(x^\gamma)$.

Corresponding, if α is the glb of numbers θ such that $A(x) = \Omega_+(x^\theta)$ we write
$A(x) = |\Omega_+(x^\alpha)$. Similarly if β is the glb of numbers θ such that $A(x) = \Omega\text{-}(x^\theta)$ we write
$A(x) = |\Omega\text{-}(x^\beta)$.

If $\alpha=\beta = \Delta$, we write $A(x) = |\Omega_{+-}(x^\Delta)$.

Let $M_1(x) = M(x)$ and for $k>1$, $M_k(x) =\sum M_{(k-1)}(n)$, where $M(x) = \sum\mu(n)$ (summation $1 \le n \le x$), where μ is the Möbius function.

Let $\Delta \in [\,½, 1]$ and let $RH(\Delta)$ be the statement that Δ is the lub of the real part of zeros of $\zeta(s)$ in the critical strip.

Seven steps to the Riemann hypothesis

- Complex analysis is a non trivial axiomatic and language extension of Peano arithmetic.
- No theorem in complex analysis leads to numerical contradiction in arithmetic.
- For $½ \le \Delta \le 1$ and each $k \ge 1$, $RH(\Delta) \equiv P(k, \Delta) \equiv \{M_k(x) = |\Omega_{+-}(x^{(k-(1-\Delta))})\}$.
- Each $P(k, \Delta)$ is logically weaker than $P(k+1, \Delta)$ in arithmetic (k=1,2,3 ……).
- Arithmetic cannot establish unbounded logically distinct propositions.
- Therefore, $RH(\Delta)$ is undecidable in arithmetic for Δ in the interval [½ , 1].
- Therefore, no zero off the line $\sigma = ½$ in the critical strip will be found by numerical investigation.

Introduction:

When a trained number theorist proves an arithmetic theorem from first principles using the rules of Peano's axioms and the various constructions which produce the familiar analytic tools needed for complex variable theory, they are in no doubt about the validity of the result. The exercise is finite and may be broken down into a number of steps. There is only a finite amount of reasoning used in the proof and as such there can only ever be a finite number of logically distinct results in the theorem. Even in inductive steps, it is recognition of form and pattern which allows generalisation and there is no unbounded logic involved.

In this discussion, real and complex analysis is viewed as a non-trivial language and axiomatic extension of arithmetic and is taken as *consistent* in that theorems of an arithmetic nature which are derived following the rules of complex variable theory will not find contradiction in arithmetic. The basis for asserting this is simply that essentially complex analysis is developed through construction of new well defined entities and the reasoning supporting the developments is inductive. We never leave a finite universe of connected thought developing the theory and the validity of the activity has been verified 'line by line' over time.

Goldfeld [1] reports on a G. H.Hardy lecture to the Copenhagen Mathematical Society:-
Hardy comments
''No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no zeros on a certain line. A proof of such a theorem, not fundamentally dependent on the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem cannot be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say 'lie deep' and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten."

History has shown this view was not entirely accurate but we know that Hardy is getting at something even if it is not quite as expressed.

The elementary proofs of the prime number theorem from the Selberg –Erdos era, overturned the view that the prime number theorem needed some theory of functions of a complex function variable and in particular the Riemann zeta function. But, we should not be drawn to reject the notion that differences in the domains of argument may shed light on the famous old problem of the Riemann hypothesis (RH) that all the non-trivial zeros of $\zeta(s)$ lie on $\sigma = ½$. Hardy was suggesting at an intuitive level that the realm of complex analysis was *necessary* in order to get some results about prime numbers.

We must keep in mind that arithmetic comes about from little more than $n \rightarrow n^*$ (the successor) and a bit of inductive argument and a meta language and it is hard to imagine we could end up with more logic at the end of the day than the logic going into the reasoning which has produced all the existing results in arithmetic. The wider realm of real and complex analysis starts talking amongst other things, about uncountable entities, tacitly assuming that there is no review needed for the 'logic' and assumptions of the new environment. In a slightly more formal sense we assume all universes are finite and reject the notion of the infinite set in this arithmetic. Thus the arithmetic and arithmetical activity to which we refer is both finite and constructive in nature. In this arithmetic, increasing knowledge in theorems is simply the construction of increasingly elaborate tautologies. With these rules, we are able to be certain of valid outcomes through finite processes.

Arithmetic and analytic definitions for the Möbius function and prime numbers

For the inductive definition of the Mobius function (the arithmetic Mobius function) in finite arithmetic we use

$\mu(1) = 1$ and with $\sum$ denoting summation $1 \le n \le N-1$, for $N>1$, $\mu(n) = 1$, if $\sum\mu(n)[N/n] = 0$, $\mu(n) = -1$, if $\sum\mu(n)[N/n] = 2$, else , $\mu(n) = 0$. This coincides with a more familiar definition of μ.

For the analytic Mobius function, we choose the definition:-

μ^*: $\sum\mu^*(n)/n^s = 1/\zeta(s)$ $(\sigma > 1)$.

We see that the $\mu^*(n) = \mu(n)$ for $1 \le n \le N$ and we drop the μ^* notation.

However, the analytic definition allows μ to be expressed in a global setting and this allows for the possibility that this setting will allow theorems about μ which are not accessible from the language and assumptions of finite inductive arithmetic. Entities like $1/\zeta(s)$ are created in the language and use ***all*** the μ values.

With the belief that the complex analysis we use, cannot produce contradictions in our arithmetic base, we thus think that a theorem derived about prime numbers in complex variable theory cannot be contradicted in arithmetic and it is not that usual to make a distinction between arithmetic and analytic theorems.

In a similar way to the μ definitions above, we have an inductive definition for primes:-

n is prime if n is not divisible by $2,3\ldots.[\sqrt{n}]$ for $n > 2$ and 2 is the first prime.

The analytic primes are of course defined by

$\sum 1/n^s = \Pi(1-1/p^{s)-1}$ $(\sigma > 1)$.

Happily, these definitions produce the same finite collections of finite primes. As with the μ function, the Riemann zeta function takes into account all the primes and we would expect some theorems to be proven which could not be proven in arithmetic. i.e. we take as read that complex analysis is both an axiomatic and language extension of arithmetic.

The non-inductive nature of the Möbius sum function M(x)

Let $M(x) = \sum\mu(n)$ (summation $1 \le n \le x$).

Let $P(\Delta)$ be the proposition: $M(x) = |O(x^\Delta)$.

If this proposition admits a direct proof (working with the Möbius function), in arithmetic for some $\Delta < 1$, then we would need some explanation of the sign changes in the values of μ behaving in such a way that the trivial estimate $M(x) = O(x^1)$ as $x\to \infty$ is reduced. The proposition $P(\Delta)$ is essentially a global property of the Möbius function and if there is not sufficient pattern in the sets { $\mu(1)$, $\mu(2)$, $\mu(3)$, ….. $\mu(N)$} for N= 1, 2 ,3 …we may expect difficulty in approaching $P(\Delta)$ directly. Thus if such an approach is impossible (for a positive result) we think of μ as being non-inductive in its function values. There is no attempt at formal definition of non-inductive here. The intuitive notion is simply – there is not the right sort of pattern in the function values to work with, in order to get a result.

We consolidate this notion of non-inductive with some observations.

Note that $M(x) = 1-\pi_1(x) +\pi_2(x)- \pi_3(x)+ \ldots$,
where $\pi_k(x)$ = the number of square free numbers less that or equal to x with exactly k prime factors.

Now, each Dirichlet series $f_k(s) = \sum(\pi_k(n)- \pi_k(n-1)\}/n^s = s\int\{\pi_k(x)/x^{(s+1)\}}dx$ (limits of integration 1 to ∞) has a singularity at s = 1.
Indeed, let $p(s) = f_1(s) = \sum 1/p^s$ where summation is over all prime numbers. Then p(s) take arbitrarily large values and small values at s=1 (Casorati-Weierstrass theorem).

With a little algebra, we see $f_k(s)$ is a polynomial in $p(s), p^2(s), \ldots ,p^k(s)$ whose coefficients involve rational numbers and finite sums of products involving p(2s), p(3s), …. .

For example :
$f_1(s) = p(s)$,
$f_2(s) = (1/2)p^2(s) – (1/2)p(2s)$,
$f_3(s) = 1/6\, p^3(s) – 1/2p(s)p(2s) + 1/3p(3s)$ …..
We see each $f_k(s)$ has an essential singularity at s=1. Indeed, the terms involving p(s) collectively become arbitrarily large in every neighbourhood of s=1, and the term which does not involve p(s) is bounded in small neighbourhoods of s=1.
Consequently, each $\pi_k(x) = |O(x^1)$ and all the $\pi_k(x)$ come into play in determining the actual order of M(x). The $\pi_k(x)$ only start contributing to M(x) from $x = p_1.p_2\ldots p_k$ (the product of the first k primes). So the task in arithmetic with a direct approach is to show there is significant cancellation between an unbounded number of different growth rates, each an $|O(x^1)$, with cancellation leading to $O(x^\Delta)$ with $\Delta < 1$.

We do note that
$1+\sum f_k(s) = \Pi(1+1/p^s) = \Pi(1-1/p^s)^{-1}/ \Pi(1-1/p^{2s})^{-1} = \sum|\mu(n)|/n^s = \zeta(s)/\zeta(2s)$
is well behaved in $\sigma > ½$ except for the simple pole at s=1. (i.e. $\sum f_k(s)$ has a simple pole at s = 1).
However, we would be asking for two quite different tricks from $p(s), p^2(s), p^3(s)$…rather than just the one. It is a cancellation between terms which is needed to get $1/\zeta(s)$ analytic for $\sigma > \Delta$ ($\Delta< 1$) and there is no simple algebraic explanation. In the $\zeta(s)/\zeta(2s)$ case, $\Delta=1$ and the explanation is found in simple algebra.

This level of difficulty is reminiscent of the formula for the prime number function $\pi(x)$ for the number of primes less than or equal to x, where for any fixed K >1
$\pi(x) = x/\ln(x) + x/\ln^2(x) + 2!x/\ln^3(x) + \ldots +(k-1)!x/\ln^k(x) + O(k!x/\ln^{(k+1)}(x))$ as $x\to\infty$.

RH is the strong asymptotic estimate $\pi(x) = li(x) + O(x^{(½ +\epsilon)})$.

Here also, we see that jump from x^1 in all terms involved to the $x^{½}$ required in the asymptotic estimate for RH to be true.

The complexity of the terms $\pi_k(x)$ hints at the non-inductive nature of the Möbius function:- not quite random function values but difficult enough perhaps in their distribution structure to make the problem of direct estimation impenetrable. There is a duality between M(x) and $\pi(x)$ in that the values of π can be derived in simple arithmetic given the values of M and vice versa. In one sense {M(1), M(2), M(3) …} and {$\pi(1), \pi(2), \pi(3)$, …} each hold exactly the same information about the primes:- the values of one set may be derived from the values in the other set and vice versa.

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Recall, RH(Δ) is the statement that Δ is the lub of the real part of zeros of $\zeta(s)$ in the critical strip.

It is well known that RH(Δ) is logically equivalent to P(Δ), and so the actual difficulty we observe with P(Δ) may well provide a pointer to the actual difficulty of proving P(Δ) by any means.
One probabilistic approach to explaining RH assumes that the sequence $\{\mu(n)\}$ is random which is quite a way from the $\{\mu(n)\}$ being completely determined. However, we see in the analytic behaviour of the $f_k(s)$ at $s = 1$, the seeds of various notions:-

- the 'random' assumption about $\{\mu(n)\}$ is not so outrageous
- a notion that the Möbius function is essentially 'non-inductive' (i.e. there is not enough in arithmetic to work in order to get an improved Δ estimate for M(x).

In the next section, we discuss a specific objection to P(Δ) being provable in arithmetic for any $½ \leq \Delta \leq 1$.

Infinite logic and the zeta function

In this discussion we look at a collection of results in the analytic theory of the Riemann zeta function ζ from the point of view of arithmetic and see that a numerical investigation which produced a zero off $\sigma = ½$ in the critical strip would imply the person finding it - capable of *'infinite'* logic – a commodity not known to reside in mere mortals.

Definitions of the infinite rightly lie in the province of the mathematician and we mean no more by infinite logic than proving an unbounded collection of logically distinct propositions or theorems.

We stand back and look at the sorts of logical phenomenon which emanate from certain mathematical activity and ponder on the consequences of the forms we observe. We observe a notion of *'infinite logic'* in the theory of the Riemann zeta function and use the rejection of this phenomenon to show that the least upper bound Δ for the real part of zeros of $\zeta(s)$ in the critical strip is undecidable in arithmetic in the interval [½, 1].

Note $\Delta = ½$ is RH and $\Delta = 1$ is equivalent to $\zeta(s)$ has zeros $s= \sigma+it$ with σ arbitrarily close to $\sigma = 1$.

We examine a collection of theorems in complex analysis which lead to difficulties in arithmetical thought.

Some basic results:

As usual let M(x) denote the Möbius sum function.

Let Δ be the lub of numbers θ such that $\zeta(s) \neq 0$ for $\sigma > \theta$. Then, it is well known that $M(x) = |O(x^{\Delta})$ and $M(x) = |\Omega_{+-}(x^{\Delta})$.

In fact the propositions $M(x) = |O(x^{\Delta})$, $M(x) = |\Omega_{+}(x^{\Delta})$ and $M(x) = |\Omega_{-}(x^{\Delta})$ are each logically equivalent to the proposition $\zeta(s) \neq 0$ for $\sigma > \Delta$, with Δ in the interval [½, 1], where Δ is defined as the lub of the real parts of zeros as above. These three results highlight the difficulty we find in elementary methods if we exclude the theory of the Riemann zeta function.
Let $M(x) = |O(x^{\Delta_1})$, $M(x) = |\Omega_{+}(x^{\Delta_2})$ and $M(x) = |\Omega_{-}(x^{\Delta_3})$. In arithmetic there is no reason to suppose

that there are relationships of any kind in $\{\Delta_1, \Delta_2, \Delta_3\}$ except $\Delta_1 = \mathrm{Max}\{\Delta_2, \Delta_3\}$ and the result $\Delta_1 = \Delta_2 = \Delta_3$ appears heavily dependent on analytical properties of $\zeta(s)$.

The classical analytical approach to knowing some things about the oscillatory nature of M(x) is via the theory of the Riemann zeta function and a theorem of Landau's that a function represented by a Dirichlet series, with real coefficients which are eventually of one sign, has a singularity at the real point on the line of convergence of the series. In simple terms – the function becomes awkward at the real point on its line of convergence.

Now,

$\sum M(n)/n^s = \zeta(s-1)/(s-1) + h_1(s)$ (summation $1 \leq n \leq \infty$)

where $h_1(s)$ is regular for $\sigma > 1$.

Knowing that ζ is zero free for $\sigma > \Delta$ and has a zero arbitrarily close to (or on) the line $\sigma = \Delta$, we may use Laundau's theorem to show that $M(x) = \Omega_{+-}(x^{(\Delta-\epsilon)})$ as $x \to \infty$.

[Consider $\sum\{M(n) + an^\theta\}/n^s$ for given $\theta < \Delta$, assuming that $M(n) + an^\theta$ is eventually of one sign. Variation on this method, quantifying the sign change in $\pi(x) - li(x)$, may be seen in Ingham [2]].

Given that $\Delta \geq ½$ we have the unconditional result $M(x) = \Omega_{+-}(x^{((1/2)-\epsilon)})$ as $x \to \infty$.

Happily, numerical investigation provides a good illustration of this property without having to carry out extensive numerical investigation.

In summary, we note key results from complex variable theory used to prove these logical relationships and the unconditional result:

- $\zeta(s)$ has one zero arbitrarily close to $\sigma = \Delta$ or on $\sigma = \Delta$ and $\Delta \geq 1/2$
- $\zeta(\sigma)$ is non zero for $\sigma > \Delta$
- Landau's theorem as above.

Landau's theorem is crucial in the Ω_{+-} results and there is no equivalent in arithmetic or real analysis.

Indeed, note that $\sum 1/n\log^2(n)$ is convergent.

e.g $\sum(1/\log^2(n))/n^s$ is not awkward at $s = 1$.

We return to Hardy's comments above and suggest that the evidence for suspecting $M(x) = \Omega_{+-}(x^{((1/2)-\epsilon)})$ as $x \to \infty$ is not available without using the analytic properties of $\zeta(s)$ is much stronger than the early suspicion about the prime number theorem.

We consolidate this increased suspicion by building up results which flow from the likes of $M(x) = \Omega_{+-}(x^{(\Delta-\epsilon)})$ as $x \to \infty$ without any additional *logic* to observe a playground in which infinite logic appears.

A simple image to understand how this can all happen is to see working in the language and assumptions of complex variable as - passing from arithmetic – through a looking glass- to a new exciting world - where strange thing may happen. The arithmetician doing this is quite happy that the journey through the looking glass is quite reversible and there is a sensible continuity in the whole experience but the passage does involve assumptions which are out of reach on the home turf of arithmetic.

Let $M_1(x) = M(x)$ and for $k>1$, $M_k(x) = \sum M_{(k-1)}(n)$

(summation $1 \leq n \leq x$).

The Ω_{+-} results for $M(x)$ ($M_1(x)$) may now be stated precisely for the $M_k(x)$:-

Theorem

Each $M_k(x) = |\Omega_{+-}(x^{(k-(1-\Delta))})$ is logically equivalent to the statement that Δ is the lub of numbers θ such that $\zeta(s) \neq 0$ for $\sigma > \theta$.

This is easily shown using induction and the following general result for Dirichlet series:-

Let $f(s) = \sum a(n)/n^s$ (summation $n \geq 1$) be convergent for $\sigma > a > 0$ and let $A(x) = \sum a(n)$ (summation $1 \leq n \leq x$).

Then $\sum A(n)/n^s = \{1/(s-1)\}f(s-1) + h(s)$ where $h(s)$ is analytic for $\sigma > a$.

The method for showing the oscillatory nature of $M(x)$ may be used to derive the general result in the theorem statement.

But the induction starts from $\sum M(n)/n^s = \zeta(s-1)/(s-1) + h_1(s)$ (summation $1 \leq n \leq \infty$)

where $h_1(s)$ is regular for $\sigma > 1$. i.e. it uses a starting theorem in the theory of functions of a complex variable in the inductive process – not some verifiable arithmetic truth.

We look at the logical structure here: For $½ \leq \theta \leq 1$, $k \geq 1$ let

Let $P(k, \Delta) \equiv \{M_k(x) = \Omega_{+-}(x^{(k-(1-\Delta))})\}$ where $\equiv$ denotes logical equivalence.

Then $RH(\Delta \equiv P(k, \Delta)$ for $k = 1,2……$.

From the point of view of arithmetic, $P(k+1, \Delta)$ is a stronger theorem than $P(k, \Delta)$ but $P(k, \Delta)$ follows trivially from $P(k+1, \Delta)$.

i.e. the oscillatory behaviour of $M_k(n)$ is not dampened in the process $M_k(x) = \sum M_{(k-1)}(n)$ (summation $1 \leq n \leq x$). The step $P(k+1, \Delta)$ provides additional logical information about μ not available in $P(k, \Delta)$.

By using information about $1/\zeta(s)$ and Landau's theorem we have found an unbounded number of logically distinct properties of μ. Thus, in a clearly defined way, $RH(\Delta)$ yeilds an infinite number of different logical properties of the Möbius function.

Since, this is the state of things for whatever the value of Δ in [½, 1] we conclude the value Δ is undecidable in [½, 1]. i.e. no value of Δ in [½, 1] is provable in arithmetic.

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Then a zero off the line $\sigma = ½$ would establish a contraction of this interval and a contradiction in arithmetic.

Consequently, no numerical investigation will locate a zero of $\zeta(s)$ in the critical strip off the line $\sigma = ½$ and RH(½) is true in the realm of complex analysis.

Notes:

(a) A corollary to the conclusion above is that in numerical investigation, zeros of $\zeta(s)$ will be found to be simple. Since RH is unprovable in arithmetic the proposition $M(x) = O(x^{((1/2)+\epsilon)})$ is unprovable in arithmetic. Then the stronger proposition $M(x) = O(\sqrt{x})$ is also unprovable. It is well known that this proposition $M(x) = O(\sqrt{x})$ implies the simplicity of the zeros of $\zeta(s)$ and hence no numerical evidence to contradict this could ever be found. i.e a located multiple zero would imply $M(x) \neq O(\sqrt{x})$.

(b) Another interesting question is whether any Ω_{+-} theorem $M(x) = \Omega_{+-}(x^{\Delta})$ with $0 < \Delta < ½$ is possible without going through the theory of the Riemann zeta function. Like the prime number theorem it may be that arithmetic will be able to stretch to $M(x)$ having unbounded sign change but at this stage no proof by elementary methods with $0 \leq \Delta < ½$ is known (author).

References

[1] Goldfeld D. http://www.math.columbia.edu/~goldfeld/ErdosSelbergDispute.pdf

[2] Ingham A.E. The Distribution of Prime Numbers, Cambridge Tracts in Mathematics and Mathematical Physics, No 30.

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