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The Lindelöf hypothesis revisited

by

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Abstract

The Lindelöf hypothesis (LH) and the Riemann hypothesis (RH) follow on from statements which are purely arithmetic in nature. The principal point of this discussion is to demonstrate that the Lindelöf hypothesis is unprovable in rational arithmetic. Consequences of this may be applied to the arithmetic of calculating consecutive zeros of the Riemann zeta function ζ in the critical strip.

Introduction

We use the words unprovability/unprovability in the context of a domain or universe of argument.

In the case of rational arithmetic (UD1) and real/complex analysis (UD2) we consider propositions which we regard as being one of true or false or unprovable within the domain.

A proposition in UD1 is unprovable in UD1 if resolution of the proposition as exactly one of true or false or unprovable involves constructions or entities whose existence necessarily requires some additional assumptions or constructions/entities from UD2.

Example:

Let $S(N) = \sum_{n \le [(\sqrt{2})N]} n$.

Prove that $S(N) = N^2 + O(N)$ as $N \to \infty$.

We assume N is always a natural number. With the notation [x]=x-{x} where [ ] denotes the greatest integer function we have

$$S(N) = N^2 - \left(2\{(\sqrt{2})N\} + 1\right)N + \{(\sqrt{2})N + 1\}\{(\sqrt{2})N\}.$$

In UD2 then we have

$S(N) = N^2 + O(N)$ as $N \to \infty$.

Is this estimate provable in UD1?

We do have a UD1 interpretation on the summation condition $n \le (\sqrt{2})N$.

i.e natural numbers n such that $n^2 \le 2N^2$. Thus the summation condition is clearly understood in UD1.

We could use this condition without mentioning $\sqrt{2}$ but interpretation does not guarantee proof.

If we let $\theta(N)$ denote the largest n satisfying $n^2 \leq 2N^2$ we cannot work with $\theta(N)$ in summation because it is not defined as a function of N in UD1. We simply cannot specify the finite collection of numbers involved in the summation as a UD1 function of N. We cannot define the summation in UD1, and hence we cannot work with the summation in UD1. When we try to think in terms of approximations to $\sqrt{2}$ we have to know it is $\sqrt{2}$ which we are approximating and this puts the proof out of UD1.

In UD1, $\quad S(N) = \frac{1}{2}\theta(N)(\theta(N)+1)$ and the best asymptotic estimate is $S(N) = O(N^2)$.

i.e. the exact asymptotic term is not provable in UD1 although we know it is true in UD2.

In a sense, $\theta(N)$ in an irrational function of N.

Section 1 –The arithmetical background

It was noted in the earlier discussion (Braun [1]) that under the assumption of the truth of the Lindelöf hypothesis the

Dirichlet series $\sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} = (1-\frac{2}{2^s})^k \zeta^k(s) \quad (k = 1,2,3 \ldots.)$ each converge for $\sigma > 1/2$.

i.e. with the truth of the Lindelöf hypothesis, we see an unbounded number of series converging for $\sigma>1/2$ whereas the half plane of absolute convergence of each series is $\sigma=1$.

Intuitively, this mountain of conditional convergence is a lot to ask of rational arithmetic (UD1).

With the distinction between UD1 and UD2 more clearly understood then the original question modifies to the form:

Is the Lindelöf hypothesis unprovable in UD1?

Would a proof that it is true or a proof that it is false necessarily involve UD2 constructions?

We do have a UD1 interpretation of the above series convergence which relates to the Lindelöf hypothesis because of the relationship between the half planes of convergence of the series and the order of the asymptotic growth of their arithmetical coefficient sums

$A_k(x) = \sum_{n \leq x} a_k(n)$ as $x \to \infty$.

Thus, because RH implies LH we have the simple line of implication: for any fixed $k > 1$,

$(A_k(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ as $x \to \infty$ unprovable) $\to$ LH unprovable $\to$ RH unprovable

where unprovable means unprovable in UD1. In particular, in UD1 we focus on the chain

$(A_2(x^2) = O(x^{1+\epsilon})$ as $x \to \infty$ unprovable) $\to$ LH unprovable $\to$ RH unprovable.

Section 2 - Focus on $A_2(x^2) = O(x^{1+\epsilon})$ as $x\to\infty$

We use ϵ in its usual role in analysis but restrict values to rational numbers.

Let D(x) denote the sum of the number of divisors of the natural numbers less than or equal to x.

Let $l(x) = \sum_{n\le x} \frac{1}{n}$.

In UD1 number theory, with rational x we have the 'sum of divisors' equation

$$D(x^2) = \sum_{n\le x^2} [\frac{x^2}{n}]$$

$$= \sum_{n\le x} [\frac{x^2}{n}] + \sum_{n>x} [\frac{x^2}{n}]$$

$$= 2\sum_{n\le x} [\frac{x^2}{n}] - [x]^2$$

$$= 2x^2 \sum_{n\le x} \frac{1}{n} - x^2 + O(x) \text{ as } x \to \infty.$$

$$\text{i.e. } D(x^2) \quad = 2x^2 l(x) - x^2 + O(x) \text{ as } x \to \infty \ldots\ldots\ldots\ldots (1).$$

See for example Gelfond and Linnik [1] pps. 53-54 or Tenenbaum [1] p. 37 and pps. 50-51.

We also have the more familiar classical asymptotic decomposition

$$D(x^2) = 2x^2 \ln(x) + (2\gamma - 1)x^2 + O(x) \text{ as } x \to \infty.$$

ln(x) is not recognised in UD1 because it is irrational for all positive rational numbers except n=1.

We necessarily need to move to the land of UD2 via the connecting bridge

$$l(y) = \ln(y) + \gamma + O\left(\frac{1}{y}\right) \text{ as } y \to \infty \ldots\ldots\ldots (2)$$

to relate the two estimates.

From the relationships between coefficient sums in Dirichlet series multiplication we see

$$A_2(x^2) \quad = D(x^2) - 4D\left(\frac{x^2}{2}\right) + 4D\left(\frac{x^2}{4}\right) \ldots\ldots (3).$$

If we use the classical UD2 estimates for D(x), there is much cancellation in the terms involving $x\ln(x)$ and γx and in the simplification of (3) and we obtain $A_2(x^2) = O(x)$ as $x \to \infty$.

We cannot contradict this result in UD1 hence in UD1 it is either true or unprovable.

In UD1 the corresponding expression for (3) using (1) is

$$A_2(x^2) = 2x^2\left\{l(x) - 2l\left(\frac{x}{\sqrt{2}}\right) + l\left(\frac{x}{2}\right)\right\} + O(x) \text{ as } x \to \infty.$$

Thus, in UD1, we require a proof that

$$\left\{l(x) - 2l\left(\frac{x}{\sqrt{2}}\right) + l\left(\frac{x}{2}\right)\right\} = O\left(\frac{1}{x^{1-\epsilon}}\right) \text{ as } x \to \infty \text{ to obtain } A_2(x^2) = O(x^{1+\epsilon}) \text{ as } x \to \infty.$$

This not an expression in UD1 as $\sqrt{2}$ is irrational and the middle sum cannot be specified in UD1.

We now move to

Theorem 1

RH is unprovable in arithmetic (UD1)

Proof

In UD1

$(A_2(x^2) = O(x^{1+\epsilon})$ as $x \to \infty$ unprovable$) \to$ LH unprovable $\to$ RH unprovable.

Thus neither RH true nor RH false can be contradicted by numerical investigation using UD2 theory.

A computer aided approach to estimating consecutive zeros will thus show zeros to lie on $\sigma=1/2$ but such an approach would not be able to conclude that a zero off the line was impossible.

The binary nature of computer programs which in themselves have no knowledge of UD2 constructs, with calculations never involving more than finite prescribed series, lead us to think of the activity as essentially fully justified in UD1 because all the UD2 theory is collapsed into UD1 interpretation. Every irrational entity in the theory is replaced by finite rational series to the level of approximation required. The difference between UD1 and UD2 as far as calculations go do not involve the differences in assumptions which allow them to be thought of as separate universes.

This ‘value’ compatibility between UD1 and UD2 allows the sensible transition from arithmetic to complex analysis in this context.

References

[1] Avigad J. Number theory and elementary arithmetic. Philosophica Mathematica (3) Vol 11 (2003), pp. 257-284 (General background)

[1]Braun P. Is the Lindelöf hypothesis decidable ? www.peterbraun.com.au

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[1] Gelfond A.O.and Linnik. Yu.V. Elementary Methods in Analytic Number Theory.
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[1] Tenenbaum G. Introduction to analytic and probabilistic number theory. Cambridge studies in advanced mathematics. 46.