# A short explanation of the Riemann hypothesis (RH)

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Let $\varphi(\mathrm{x}) = \sum_{\substack{\mathrm{p}\le \mathrm{x}\\ \mathrm{p\ prime}}} \ln(\mathrm{p})$, $\pi(\mathrm{x}) = \sum_{\substack{\mathrm{p}\le \mathrm{x}\\ \mathrm{p\ prime}}} 1$ and $\mathrm{l}(\mathrm{x}) = \sum_{1\le \mathrm{n}\le \mathrm{x}} \frac{1}{\mathrm{n}}$.

Let $\Theta = \mathrm{lub}\{\Delta\colon \zeta(\sigma + \mathrm{it}) \neq 0 \text{ for } \sigma > \Delta\}$ and $|\mathrm{O}\left(\mathrm{x}^{\delta}\right)$ mean $\delta = \mathrm{lub}\{\Delta\colon \mathrm{E(x)}=\mathrm{O}(\mathrm{x}^{\Delta})\}$.

It is well known that

$\varphi(\mathrm{x}) = \mathrm{x} + \mathrm{E}(\mathrm{x})$ where $\mathrm{E}(\mathrm{x}) = |\mathrm{O}\left(\mathrm{x}^{\Theta}\right)$ and $1/2 \le \Theta \le 1$.

Consequently

$$\sum_{\substack{\mathrm{p}\le \mathrm{x}\\ \mathrm{p\ prime}}} \mathrm{l}(\mathrm{p}) = \mathrm{x} - \gamma\pi(\mathrm{x}) + \mathrm{E}_1(\mathrm{x}) \text{ where } \mathrm{E}_1(\mathrm{x}) = |\mathrm{O}\left(\mathrm{x}^{\Theta}\right), \text{ where } \gamma \text{ is Euler's constant.}$$

Since we cannot prove $\gamma$ exists in UD1 the best possible asymptotic estimate for $\mathrm{E}_1(\mathrm{x})$ in UD1 is $\mathrm{E}_1(\mathrm{x}) = \mathrm{O}(\pi(\mathrm{x}))$ and so $\mathrm{E}_1(\mathrm{x}) = \mathrm{O}(\mathrm{x})$ is the best possible x power asymptotic estimate in UD1.

Consequently, any choice of $\Theta$ in the range $1/2 \le \Theta \le 1$ cannot be contradicted in UD1 by numerical computer investigation using UD2 theory because the specific mathematical content in the UD2 theory needed to calculate zeros may be argued inductively from UD1. A zero off the line $\sigma=1/2$ would contradict the unprovability.

Calculated zeros will thus lie on $\sigma=1/2$.

## Notes

1. Real and complex analysis (UD2) and arithmetic (UD1) may be viewed as independent axiomatic systems because UD2 requires an additional enabling assumption:-
   the equivalences class formed by the notion of 'Cauchy convergence' is uncountable in terms of class membership. Amongst other things from the inductive point of view of arithmetic the independence of the binary operations from class representation requires an assumption beyond arithmetic. We make assertions about uncountable classes of elements beyond the reach of induction in UD1. The construction of UL2 requires an assumption about the sensibility of this.
2. Compatibility between UD1 and UD2 is reflected in the fact that no numerical contradiction is possible in the derivation of numerical results.
   i.e. the theorems of UD2 in so far as they related to numerical things may be argued out with

the weak induction of UD1 even though a rigorous proof requires the additional assumptions of UD2.

3. An inductive proof in UD2 about a numerical result may not be provable in UD1.
e.g. any rational convergent sequence where the limit is a known irrational.
The exact unbounded oscillatory behaviours of the Möbius sum function and the higher sum functions is an example which has been discussed at length..
A simple observation to distinguish UD1 from UD2 is to note that one of the main bridges between UD1 and UD2 is the identity:-

$$x = \exp(\ln(x)).$$

No rational part of this except x=1 is in UD1 since both exp(x) and ln(x) are irrational for rational argument. The next hurdle in UD1 as noted in various places is the equally remarkable

$$l(x) = \ln(x) + \gamma + O\left(\frac{1}{x}\right).$$

Also note that

$$O\left(x^{\Delta}\right) \text{ as } x \to \infty$$

has a distinct logarithmic look for irrational numbers Δ in UD1 even though is has interpretation in UD1 for rational numbers. In this RH context we are looking for logarithmic type estimates for functions in UD1 and it is then not too surprising we come up against a brick wall.

4. We really need to understand the difference between the 'concrete' theory around the Riemann zeta function in prime number theory, the fundamental difference between inductive arithmetic and the un-countability of the continuum and what we are doing in numerical investigations –computer aided – into the non-trivial zeros of zeta. The fact that we are able to stepwise compute zeros of ζ is really a proof that the results are gained inductively from UD1.
In the development of the theory in UD2 we deal with prescribed series and sequences and the properties we use are derived using inductive arguments in UD1. We work in a miniscule part of UD2 which has UD1 interpretation. As such we do not need to worry about the theoretical difference between UD1 and UD2. Thus when we are looking for zeros numerically we are to all intents and purposes still in UD1 – we cannot contradict the UD1 truth of unprovability.
Thus, the behaviour of some pure mathematicians to not give a beggar about consistency and completeness in the formal universe of philosophical and logical thought has a kind of soundness for certain sorts of numerical problems contained within UL2.

5. Another interesting proof of RH comes about directly from establishing that
$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$ is not provable in arithmetic (UD1) (μ denotes the Möbius function).

$\mathrm{M}(\mathrm{x}) = \sum_{1 \leq n \leq N} \mu(n) = O(\sqrt{x})$ is also unprovable in UD1 and hence all computed zeros will be simple.

**References:**


The original discussion 'The Euler-Mascheroni constant and Riemann's hypothesis' recorded on www.peterbraun.com.au and also posted on the website of M. Watkins at www.maths.ex.ac.uk/~**mwatkins**/zeta/RHproofs.htm provides detail for separating UD1 and UD2. Jeremy Avigad's paper 'Number Theory and Elementary Arithmetic', Philosophia Mathematica Vol 11 (2003) pp.257-284 may encourage focus on a foundational approach to RH.