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Off-line zeros of the Riemann zeta-function
by
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Note the algebraic identity

$$\sum_{n\le N} a(n)A(n) = \frac{1}{2}\, A^2(N) + \frac{1}{2}\sum_{n\le N} a^2(n), \quad \text{where } A(N) = \sum_{n\le N} a(n).$$

With the choice $a(n) = \frac{(-1)^{n+1}}{n^s}$, we thus have

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^s} = \frac{1}{2}\{\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}\}^2 + \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n^{2s}} \ldots\ldots\ldots\ldots..(1).$$

Point-wise, for $\sigma > 0$,

$$\lim_{N\to\infty} \{\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}\}^2 = \{1 - \frac{2}{2^s}\}^2 \zeta^2(s),$$

Also, for $\sigma > 1/2$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^{2s}} = \zeta(2s).$$

Thus, for $\sigma > 1/2$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^s} = \frac{1}{2}\{1 - \frac{2}{2^s}\}^2 \zeta^2(s) + \frac{1}{2}\zeta(2s).$$

Using the well-known theorem on Dirichlet series half planes of convergence, since for $\sigma > 0$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^s} = \frac{1}{2}\{1 - \frac{2}{2^s}\}^2 \zeta^2(s) + \frac{1}{2}\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^{2s}},$$

we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^s} \text{ does not exist at any point in } 0 < \sigma < \frac{1}{2} \ldots..(1).$$

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On the other if ζ had a zero in $\sigma>1/2$, $s=1-\Delta+it_\Delta$ say, we may raise the possibility that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^{\Delta+it_\Delta}}\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r^{\Delta+it_\Delta}}=\frac{1}{2}\zeta(2\Delta+it_\Delta),\text{ which would contradict (1) .}$$

The assumption of a zero provides possible conditions for convergence in that the inner sum tends to zero as $n\to\infty$.

We may express this as

Conjecture:

If $0<\Delta<1/2$ and $\zeta(\Delta+it_\Delta)=0$ then

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^{\Delta+it_\Delta}}\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r^{\Delta+it_\Delta}}=\frac{1}{2}\zeta(2\Delta+it_\Delta)\ .$$

Corollary to conjecture:

RH.

Notes

If we consider any one of the uncountable number of Dirichlet series of the form

$L_\delta(s)=\sum_{n\geq1}\frac{\delta(n)}{n^s}$, where $\delta(n)\in\{-1,1\}$ we have

$$\sum_{n=1}^{\infty}\frac{\delta(n)}{n^s}\sum_{r=1}^{n}\frac{\delta(r)}{r^s}-\frac{1}{2}\{\sum_{n=1}^{\infty}\frac{\delta(n)}{n^s}\}^2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{2s}}\quad(\sigma>1)$$

which shows up ζ as the common ‘residue’ of an uncountable number of processes.

The prime decomposition of all the analytic numbers in order is also an uncountable feat for arithmetic.