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$$\prod_{a<n} 2\sin\left(\frac{\pi a}{n}\right) = n \text{ and } \prod_{a<n} 2\sin\left(\frac{\pi a}{2n}\right) = \sqrt{n} \qquad (n > 1)$$

by Peter Braun (* See notes below)

These two formulae seem somewhat neglected in elementary texts compared to corresponding results involving just Farey fractions in the sin argument.
In an age of easily usable spreadsheets the 'magical' nature of these results is easily verified in particular cases provided the approximation to π is good enough.

Let T(n) denote the collection of nth primitive roots of unity and correspondingly let h(n) denote the collection of Farey fractions {a/n : (a,n) =1, a ≤ n}.

T(n) = { cos(2πq) +i sin(2πq) : q Є h(n)}.

Kluyer's formula for the Ramanjuan sum $c_n(k)$ – the sum of the k^{th} powers of the primitive n^{th} roots of unity - is

$$\sum_{\xi \in T(n)} \xi^k = \sum_{g|(n,k)} \mu\left(\frac{n}{g}\right) g.$$

Hence

$$\sum_{\xi \in T(n)} \log(1-\xi) = \sum_{k=1}^{\infty} \sum_{g|(n,k)} \{\mu\left(\frac{n}{g}\right) g\}/k\,.$$

Note that a generating function for the Ramanjuan sums $c_n(k)$ is

$$\sum_{g|n} \{\mu\left(\frac{n}{g}\right) g\}/g^s \sum_{m=1}^{\infty} 1/m^s = \sum_{k=1}^{\infty} \sum_{g|(n,k)} \{\mu\left(\frac{n}{g}\right) g\}\ /k^s\,.$$

Indeed, in the LHS multiplication of Dirichlet series, a coefficient term μ(n/g)g contributes additively to the coefficient of $1/k^s$ on the RHS if and only if g|n and g|k. This is equivalent to g|(n,k) which then provides the form of the RHS.

Hence

$$\lim_{s\to 1+} \sum_{g|n} \{\mu\left(\frac{n}{g}\right) g\}/g^s \sum_{m=1}^{\infty} 1/m^s = \lim_{s\to 1+} \sum_{k=1}^{\infty} \sum_{g|(n,k)} \{\mu\left(\frac{n}{g}\right) g\}\ /k^s\ .$$

That is

$$\sum_{g|n} \mu\left(\frac{n}{g}\right) \log(g) = \sum_{\xi \in T(n)} \log(1-\xi)\ .$$

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In other words

$$\Lambda(\mathrm{n}) = \sum_{\xi \in \mathrm{T}(\mathrm{n})} \log(1-\xi) \ \ .$$

These formulae are stated in the Wikipedia section on Ramanujan sums,
http://en.wikipedia.org/wiki/Ramanujan's_sum

Thus

$$\Lambda(\mathrm{n}) = \sum_{\xi \in \mathrm{T}(\mathrm{n})} \mathrm{Re}\{\log(1-\xi)\},$$

or, in other words,

$$\Lambda(\mathrm{n}) = \sum_{\mathrm{a} \in \mathrm{h}(\mathrm{n})} \log\left(2\sin\left(\frac{\pi \mathrm{a}}{\mathrm{n}}\right)\right) .$$

Finally, note that

$$\prod_{\mathrm{a}<n} 2\sin\left(\frac{\pi \mathrm{a}}{\mathrm{n}}\right) = \prod_{\mathrm{g}|\mathrm{n}} \prod_{\mathrm{a} \in \mathrm{h}(\mathrm{g})} 2\sin\left(\frac{\pi \mathrm{a}}{\mathrm{g}}\right).$$

Hence

$$\prod_{\mathrm{a}<n} 2\sin\left(\frac{\pi \mathrm{a}}{\mathrm{n}}\right) = \prod_{\mathrm{g}|\mathrm{n}} \mathrm{e}^{\Lambda(\mathrm{g})}.$$

i.e.

$$\prod_{\mathrm{a}<n} 2\sin\left(\frac{\pi \mathrm{a}}{\mathrm{n}}\right) = \mathrm{n}.$$

Corollary 1

For $n > 1$

$$\prod_{a<n} 2\sin\left(\frac{\pi a}{2n}\right) = \sqrt{n}.$$

For n odd:

$$\prod_{a\le(n-1)/2} 2\sin\left(\frac{\pi a}{n}\right) = \sqrt{n}.$$

For n even:

$$\prod_{a\le(\frac{n}{2})-1)} 2\sin\left(\frac{\pi a}{n}\right) = \left(\sqrt{\frac{n}{2}}\right).$$

Notes:

I recently came across a technique in some old notes from A Zulauf which provides a quicker and elementary proof of the first formula in the title:

$$\prod_{a=1}^{n-1}\left(1 - e^{\frac{2\pi i a}{n}}\right) = \lim_{t\to1} \frac{1}{t-1}\prod_{a=1}^{n}(t - e^{\frac{2\pi i a}{n}}) = \lim_{t\to1} \frac{1}{t-1}(t^n - 1) = n.$$

We easily see $\mathrm{Re}\{\log\left(1 - e^{\frac{2\pi i a}{n}}\right)\} = \log(2\sin\left(\frac{\pi a}{n}\right)) \qquad (1 \le a < n),$

and the result follows.