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**The Liouville function on Farey fractions and the Riemann hypothesis**
**by**
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Connections between the Riemann hypothesis (RH) and the distribution of ordered Farey fractions have been discussed since the time of Landau and Franel. The basic results and some more recent observations are included in Zulauf [3]. Additional developments and references are provided in Kanemitsu and Yoshimoto [1].
The purpose of this note is to examine logical equivalences between the growth rate of the sum of the Liouville function values of Farey fractions and the Riemann hypothesis. Thus the Farey fractions are closely connected to the Riemann hypothesis (RH) both in terms of their distribution on [0,1] and also in terms of their multiplicative structure as valued by Liouville's function.

We abbreviate $\forall\, \epsilon > 0\colon f(n) = O\left(n^{\Delta+\epsilon}\right)$ as $n \to \infty$ to $f(n) = O\left(n^{\Delta+\epsilon}\right)$ throughout.

Let $H_N$ denote the set of Farey fractions of order N. i. e. $H_N = \left\{\frac{a}{b} : 1 \le a \le b \le N, (a, b) = 1\right\}$.

and for $n \ge 1$ let $h_n = \left\{\frac{a}{n} : 1 \le a \le n, (a, n) = 1\right\}$.

Let $M(N) = \sum_{n \le N} \mu(n)$ and $S(N) = \sum_{n \le N} \lambda(n)$ where $\mu$ denotes the Möbius function and

$\lambda$ denotes Liouville's function. We assume two quite well known equivalences to RH. Namely

$RH \equiv \left[M(N) = O\left(N^{\frac{1}{2}+\epsilon}\right)\right] \equiv \left[S(N) = O\left(N^{\frac{1}{2}+\epsilon}\right)\right]$, Edwards[2], Titchmarsh [4].

We extend the definition of $\lambda$ to positive fractions using $\lambda\left(\frac{a}{n}\right) = \frac{\lambda(a)}{\lambda(n)}$ and note this definition is

independent of the natural numbers a and n defining a particular fraction.

Let $h(n) = \sum_{q \in h_n} \lambda(q)$ and let $H(N) = \sum_{q \in H_N} \lambda(q)$. Clearly $H(N) = \sum_{n \le N} h(n)$.

**Proposition:**

$$RH \equiv \; h(N) = O\left(N^{\frac{1}{2}+\epsilon}\right) \equiv \quad H(N) = O(N^{1+\epsilon})\,.$$

**Proof:**

We note

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(1) $H(N) = \sum_{1\le n\le N} M(\frac{N}{n}) \sum_{1\le a\le n} \lambda\left(\frac{a}{n}\right).$

Indeed,

the terms contributing to $\lambda(q) = \lambda\left(\frac{a}{n}\right)$ where $(a, n) = 1$ and n is fixed are $M\left(\frac{N}{rn}\right)$ for $1 \le r \le \left[\frac{N}{n}\right]$,

and these sum to unity for each n with $1 \le n \le N$.

Variations on formula (1) date to at least Landau and Franel (Edwards [2]).

Consequently,

(2) $H(N) = \sum_{1\le n\le N} M\left(\frac{N}{n}\right)\lambda(n)S(n)$ , and interpreting this in terms of Dirichlet series,

(3) $\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \left\{\frac{1}{\zeta(s)}\right\}\sum_{n=1}^{\infty} \frac{\lambda(n)S(n)}{n^s}$ ($\sigma > 2$ where as usual $s = \sigma + it$).

We see from (3) that $h(n) = \sum_{g|n} \mu(g)\lambda\left(\frac{n}{g}\right)S\left(\frac{n}{g}\right)$ and consequently $|h(n)| < d(n)\max_{g|n}\left|S\left(\frac{n}{g}\right)\right|$,

where $d(n)$ denotes the number of divisors of number. Using $d(n) = O(n^{\epsilon})$ and,

assuming RH, the well known estimate $S(n) = O(n^{\frac{1}{2}+\epsilon})$, we have

$|h(n)| = O(n^{\frac{1}{2}+\epsilon})$.

Conversely, assuming $h(n) = O\left(n^{\frac{1}{2}+\epsilon}\right)$ and using the Mobius inversion formula and (3)

we have $\lambda(n)S(n) = \sum_{g|n} h(g)$ and consequently $S(n) = O(n^{\frac{1}{2}+\epsilon})$ and so RH is true.

We now show RH $\equiv H(N) = O(N^{1+\epsilon})$.

From the two ways of expressing the coefficient sum in a product of Dirichlet series, from (2) we also have

(4) $H(N) = \frac{1}{2}\sum_{n\le N} \mu(n)S^2\left(\frac{N}{n}\right) + \frac{1}{2}\sum_{n\le N} \mu(n)\left[\frac{N}{n}\right].$

If RH is true we have the well known estimate $S(N) = O\left(N^{\frac{1}{2}+\epsilon}\right)$, Edwards [2] and hence,

$H(N) = O(N^{1+\epsilon})$.

Conversely, suppose $H(N) = O(N^{1+\epsilon})$. From (3) for $\sigma > 2$,

(5) $$\zeta(s)\sum_{n=1}^{\infty}\frac{h(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\lambda(n)S(n)}{n^s}.$$

Then since $\sum_{n\le N} H\left(\frac{N}{n}\right) = O(N^{1+\epsilon})$, the product of the two series on the LHS of (5) is convergent as a Dirichlet series for $\sigma > 1$, and hence $\sum_{n\le N}\lambda(n)S(n) = \frac{1}{2}S^2(N) + \frac{1}{2}N = O(N^{1+\epsilon})$.

Then $S(N) = O\left(N^{\frac{1}{2}+\epsilon}\right)$ and consequently $\zeta(2s)/\zeta(s)$ is analytic for $\sigma > 1/2$. Then RH is true.

**Notes:**

$H(N) = O(N^{1+\epsilon})$ is a stronger than $h(N) = O\left(N^{\frac{1}{2}+\epsilon}\right)$ yet both are equivalent to RH.

**References**

[1] Kanemitsu S. and Yoshimoto M. Farey series and the Riemann hypothesis. Acta Arithmetica LXXV.4. 1996

[2] Edwards H.M. Riemann's Zeta Function. Dover Publications. 2001.

[3] Zulauf. A. The distribution of Farey numbers. Journal für die reine und angewandte Mathematik, Band 289, 1977.

[4] Titchmarsh E.C. the Theory of the Riemann Zeta-function, 2nd edition, Oxford Science Publications, 1986.