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The Riemann hypothesis is undecidable in arithmetic (ii)

by

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Abstract:

The fundamental realisation mentioned in Braun [3] it that the Möbius function values generate the prime number counting function values in arithmetic. In the limit the prime number function generates the natural logarithm values on natural numbers. The limit Möbius function values thus generate the natural logarithm values on natural numbers.
The purpose of this discussion is to explain how the natural logarithm function (ln) is a key to understanding the Riemann hypothesis in arithmetic. Without the logarithm function we cannot say much about prime distribution and this leads to a proof that RH is undecidable in arithmetic. In contradiction to this, an offline zero of zeta in the critical strip could in principle be calculated in arithmetic.

Section 1

Definitions, conventions

Let

$$[x] = \sum_{\substack{n\le x \\ n \text{ natural} \\ \text{number}}} 1\ ,\quad \pi(x) = \sum_{\substack{p\le x \\ p \text{ prime}}} 1\ ,\qquad M(x) = \sum_{n\le x}\mu(n),\quad g(x) = \sum_{n\le x}\frac{\mu(n)}{n}\quad (\mu = \text{Möbius function}),$$

$$S(x) = \sum_{n\le x}\lambda(n),\quad h(x) = \sum_{n\le x}\frac{\lambda(n)}{n}\quad (\lambda = \text{Liouville's function})\ \text{ and } l(x) = \sum_{n\le x}\frac{1}{n}\ .$$

Let p, q be coprime natural numbers with p≤q.

$$A(x) = O\left(x^{\frac{p}{q}}\right) \text{ as } x\to\infty \text{ and } \sum_{n\le x^{\frac{p}{q}}} a(n) = O\left(x^{\frac{p}{q}}\right) \text{ as } x\to\infty \text{ have definitions within arithmetic.}$$

Indeed, we do not need to recognise constructions in which the existence of $x^{p/q}$ is established as we substitute the largest number k(x) such that $k(x)^p \le x^q$ in arithmetic calculations.

If $\Delta=\text{lub}\{\sigma: A(x)=O(x^\sigma)$ as $x\to\infty$, we call Δ the order of A(x) and write $A(x)=|O(x^\Delta)$ as $x\to\infty$.

Introduction

Existing prime number theory involving prime structure in natural numbers provides a wealth of indication of how little is achievable in distribution questions without using the natural logarithm and the realm of real/complex analysis.

For example, let A(n), B(n) take positive increasing integer values as n =1,2 3…. , with positive non-zero order less than unity:-
We appear to have a 'critical gap' in rational arithmetic, for example, proving that
(i) $\exists$ A(n) such that there exists a prime between n and n+A(n) for each natural number n, or
(ii) $\exists$ B(n) such that $M(n) = \Omega_{\pm}(B(n))$ as $n \to \infty$.

For A(n) the weakest case is an order less than unity and for B(n) the weakest case is an order greater than zero.
In complex analysis we know the order of such a B(n) is actually greater than or equal to ½ and (more or less) the order of such an A(x) is less than or equal to ½.

Without the logarithm the ordering of the natural numbers using their prime structure becomes a task of ever increasing difficulty. We need to discuss the exact value of primes in some theoretical way to understand the order of prime products and it is the natural logarithm which provides exact values which can be translated into the exact prime values. These logarithm values are not values in rational arithmetic (Hardy and Wright [1]).
In fact, theoretical ordering of the natural numbers via their prime structure is impossible in rational arithmetic. Indeed, allowing a bit of hand waving – to know the ordering of the numbers via their prime structure is to know ln(n) for each n and this puts us outside of rational arithmetic and the arithmetic primes. We tighten this argument up in the course of this discussion.
The principle aim of the discourse however is to show that the true propositions

$\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to\infty} \frac{M(x)}{x} = 0$, necessarily require a domain beyond rational arithmetic

for proof.

It then follows quite quickly that RH cannot be contradicted in numerical analysis by calculating stepwise the zeros of ζ in the critical strip.

Section 1

Preliminaries

By rational arithmetic we mean arithmetic based on the standard construction of the rational numbers. A rule on the arguments in proofs in this domain is they do not reference the exact values of irrational numbers or numbers whose rational status is unprovable in rational arithmetic, either implicitly or explicitly. We use UD1 (universe of discourse) to denote this argument realm. We use UD2 to denote arguments which include UD1 and the standard constructions for the real and complex numbers. As axiomatic systems, UD2 requires more assumptions than UD1 and the two may be held at arms length from each other or UD2 may be seen as a sensible extension of UD1 given certain restrictions.
In numerical work using prescribed series which are essentially based on inductive definition the two systems are consistent in that no numerical contradictions arise. For convenience we call this type of activity 'concrete theory'.

There is a reversal in terms of imagery in considering the relationship between UD1 and UD2 with regard to logic and mathematical output:-
UD1$\subset$UD2 (mathematical output) and UD2$\subset$UD1 (logic).

Simply put, UD2 has more concepts, constructions and so on and with more words there is more to write about and think about in terms of old and new problem solving.
On the other hand UD2 is born from arithmetic using additional logical assumptions and the status of proof in arithmetic and extensions of arithmetic where numerical results are involved depend on having interpretation of the proof in UD1.

If we restrict UD2 to concrete theory we have UD1=UD2 because we have the inductive link of series approximation which allows numerical arithmetic interpretation.

An important point is that with this restriction we don't need to reject our knowledge of known results in UD2 when we wish to restrict argument to UD1 because we will always have numerical (value) interpretation with no possibility of inconsistency – the construction by prescribed convergent series ensures this. We repeat that the only difference we insist on is that in UD1 proof we do not either implicitly or explicitly acknowledge the precise values of irrational numbers or numbers with unknown rational status. UD2 irrational values have approximate value in UD1.
The simple distinction in this context is between a number and the value of a number.
e.g. $\sqrt{2}$ exists in UD2 and its value is exact in UD2 but in translation in UD1 we are only able to approximate the value in numerical terms. Number and value have the same meaning for number entities in UD1 and meaning is defined inductively. Value is defined via ordering.

Thus, if we calculate step wise the zeros of ζ in the critical strip, aided by 0-1 type computer output and the concrete theory we have an exercise in arithmetic which legitimately interprets numerically the UD2 theory.

If then we establish that RH is unprovable in arithmetic we will not be able to find an offline zero in the critical strip through calculation. i.e. all calculated zeros will lie on $\sigma=1/2$. Otherwise, the numerical evidence would imply an arithmetical calculation disproving RH.
In this case we may say that RH cannot be contradicted in terms of inductive argument in UD1.
We call this the weak Riemann hypothesis because there remains at this stage the possibility that a proof involving more assumptions than the assumptions of UD1 would be able to argue the limit case that all the zeros of zeta in the critical strip lie on $\sigma=1/2$.

Section 2

Dirichlet series in the complex plane

Throughout this discussion we focus on Dirichlet series which have rational constant coefficients on numbers of the same prime structure:

i.e Dirichlet series which are rearrangements of

$$\text{i.e. } a_1 + a_2 \sum_{\substack{p \\ p \text{ prime}}} \frac{1}{p^s} + a_3 \sum_{\substack{p,q \\ p,q \text{ prime} \\ p \neq q}} \frac{1}{(pq)^s} + a_4 \sum_{\substack{p \\ p \text{ prime}}} \frac{1}{p^{2s}} + \cdots = \sum_{n \geq 1} \frac{b_a(n)}{n^s},$$

with $a_2 \neq 0$ with σ=1 defining the half plane of convergence of the series. The forms may be ordered so that a={a_i} is well defined (Braun [2], [3]).

These criteria covers many Dirichlet series associated with RH type problems, and include many of the ones which include the familiar products involving primes and prime powers with constant rational coefficients on each prime power.

We note for series of this sort there is no known proof of a convergent series having a wider half plane of convergence than σ=1.

i.e. there is no known UD1 argument that there exist rational coefficients a_i such that

$B_a(N) = \sum_{n \leq N} b_a(n)$ has order less than unity.

We know from the theory of Dirichlet series (Titchmarsh [1]) that the maximum difference between the half plane of absolute convergence and the half plane of convergence is unity for general convergent simple Dirichlet series.

We confirm in the course of the discussion that in UD1 it is not possible to tease out a difference between the half plane of absolute convergence and the half plane of convergence for the special series under consideration.

Section 3

Multiplicative functions as coding functions

Multiplicative function values may be thought of as code derived from the multiplicative structure of numbers. The Möbius function may be defined by $\mu(n) = (-1)^k$ if n is a product of k distinct primes and with function value zero otherwise. Thus

$M(x) = 1 - \pi_1(x) + \pi_2(x) - \pi_3(x) + \cdots \ldots$

where $\pi_r(x)$ is the count of square free numbers less than or equal to x with exactly r prime factors.

Other definitions for the Möbius and Liouville function are:

For $N \geq 1, \sum_{n \leq N} \mu(n)[\frac{N}{n}] = 1$ and $\sum_{n \leq N} \lambda(n)[\frac{N}{n}] = [\sqrt{N}]$.

The second formula here uses the largely irrational number √N rounded down to the nearest natural number. We do not have a simple arithmetic expression for this number in UD1 but we do have an arithmetic interpretation as k(N) where k(N) is the largest number satisfying $k(N)^2 = N$.

With this interpretation we see the formula as comprehensible in arithmetic for the inductive numbers. However, if we are dealing with limit properties involving λ such as $h(\infty)=0$, we have an element of 'unbounded verification' which is not inductive and would suggest that $h(\infty) = 0$ is a proposition outside of UD1. This is only mentioned here as a possible subject for further thought.

These last definitions may be recognised as the Dirichlet series partial coefficient sums in both sides of

$$\sum_{n\geq 1}\frac{\mu(n)}{n^s}\zeta(s) = 1 \text{ and } \sum_{n\geq 1}\frac{\lambda(n)}{n^s}\zeta(s) = \zeta(2s)$$

where ζ is Riemann's zeta function and s is the complex variable $s=\sigma+it$.

Consequently, these two relationships may be seen as special cases of the collection

$$\sum_{n\geq 1}\frac{a_k(n)}{n^s}\zeta(s) = \zeta(ks)\ (k = 1,2,3 \ldots\ldots)$$

with the Möbius function as the limit relationship as $k\to\infty$.

The comments above about the Liouville function also apply to the arithmetic functions a_k in the preceding equation and we similarly may question if any one of the equations

$$g_k(\infty) = \sum_{n\geq 1}\frac{a_k(n)}{n} = 0$$

is provable in UD1.

On the face of it the coding in these arithmetic functions is not giving much information about the distribution of prime structure of the natural numbers in arithmetic. Remarkably however the ordered values of each of these functions allows a theoretical stepwise ordering of the natural numbers by their prime structure as we see in the μ case in section 4. The ordered coding in μ values allows the recovery of the order of the natural numbers in terms of their prime structure.

We see the ordered collection of values of these functions is complicated indeed and arithmetic cannot reach the limit collection without implicitly assuming the ordering of the natural numbers by their prime structure and this puts the task in UD2.

Clearly we may sit down with a pencil and paper and start off the chain

$$1 < p_1 < p_2 < p_1^2 < p_3 < p_1p_2 < p_4 < p_1^3 < p_2^2 < \cdots \qquad (p_i = i^{\text{th}} \text{ prime})$$

and we are starting to code in the ordering of the prime structures.

This chain extends in principle indefinitely but the completion of this chain requires unbounded information which takes us out of the realm of rational arithmetic into the realm of analysis.

Indeed, armed with the natural logarithm and the observations

$2^{\alpha(n_p)} < p^n < 2^{\alpha(n_p)+1}$ and that the sequence $\{\frac{\alpha(n_p)}{n}\}$ is Cauchy convergent $\left(=\frac{\ln(p)}{\ln(2)}\right)$,

we are able to resolve the ordering of the natural numbers according to prime structure. i.e.

$u_1^{a_1}u_2^{a_2} \dots u_r^{a_r} < v_1^{b_1}v_2^{b_2} \dots v_s^{b_s}$ iff $\sum_{i\le r} a_i \ln(u_i) < \sum_{i\le s} b_i \ln(v_i)$.

Thus, at a theoretical level with the notion of limit sets in the real field we resolve the ordering of all the factorisations of numbers whereas in arithmetic with our pencil and paper we are left labouring indefinitely on an unbounded mountain which cannot be climbed.

We do not have another logically distinct theoretical way of resolving through theory in arithmetic all the decisions

$$u_1^{a_1}u_2^{a_2} \dots u_r^{a_r} < v_1^{b_1}v_2^{b_2} \dots v_s^{b_s}, \qquad u_1^{a_1}u_2^{a_2} \dots u_r^{a_r} > v_1^{b_1}v_2^{b_2} \dots v_s^{b_s}, \qquad u_1^{a_1}u_2^{a_2} \dots u_r^{a_r} = v_1^{b_1}v_2^{b_2} \dots v_s^{b_s}\ .$$

We know the values of the natural logarithm are irrational and so in theory expressions resolution of the ordering of natural by their prime structure puts us necessarily in UD2.

Yet the natural logarithm is still quite a blunt instrument in this context.

Indeed, we know from the painstaking work of the elementary methods in number theory and the above observation that to get a non trivial estimate for $M(x)$ we need to inject something about prime structure into the argument. The work in elementary methods proving the prime number theorem in the form $M(x)=o(x)$ as $x\to\infty$ makes extensive use of the natural logarithm, providing

$|M(x)|\ln(x) \le \sum_{n\le x}\left|M\left(\frac{x}{n}\right)\right| + O(x\ln(\ln(x))$ as $x \to \infty$, and this implies the required result.
See for example Gelfond and Linnik [1].

Human ingenuity manages to eek out the theorem $M(x)=o(x)$ as $x\to\infty$ using this equation. We reiterate though that this is a proof in UD2 as the logarithm is persona non grata in UD1.

Section 4

Mixing up arithmetic and analysis

We have asserted above that the ordered μ values allow recovery of the ordered natural numbers in terms of their prime structure. The relevance of this is that it puts the task of proving limit properties in the convergences of certain conditionally convergent series and some asymptotic estimates outside of the rational domain.

The overall structure from hereon is to discuss a meaning for the logical chain

$\{\boldsymbol{\mu}(\mathbf{n}): \mathbf{n} \ge \mathbf{1}\} \to \{\boldsymbol{\pi}(\mathbf{n}): \mathbf{n} \ge \mathbf{1}\} \to \{\mathbf{ln}(\mathbf{n}) : \mathbf{n} \ge \mathbf{1}\}$

and then extend it to explain

$\{\boldsymbol{\mu}(\mathbf{n}): \mathbf{n} \ge \mathbf{1}\} \to \{\boldsymbol{\pi}(\mathbf{n}): \mathbf{n} \ge \mathbf{1}\} \to \{\mathbf{ln}(\mathbf{n}) : \mathbf{n} \ge \mathbf{1}\} \to$ **RH unprovable in UD1**.

Before proceeding to the main argument we draw a distinction between purely arithmetic functions and those which also appear in analytical work.

With the clear separation between arithmetic (UD1) and analysis (UD2) ([1]) we may make a distinction between arithmetic (UD1) numbers, primes, number functions and so on and the corresponding analytic (UD2) entities. Where no limit properties are involved and collections are finite the relevant corresponding entities will coincide.

The arithmetic primes are defined inductively in the broad sense and this is implicit in the argument that the prime number sequence is unbounded. The unique factorisation of finite numbers into prime factors is also provable by mathematical induction in UD1 (Davenport [1]).

The analytic primes are defined by a function not connected in an exact numerical way to arithmetic by

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_{p\ \mathrm{prime}} (1-\frac{1}{p^s})^{-1} = \prod_{n\geq 1}(1-\frac{1}{p_n^s})^{-1}\ (\sigma > 1), \text{where the primes}$$

$p_1, p_2, p_3, \ldots$ are in ascending order. Indeed, as we have noted, the act of ordering all the prime products to provide the definition for $\zeta(s)$ implies defining $\ln(n)$ for each number n and this is not possible in UD1 which only has rational values in that axiomatic number system.
Transition between the precise UD2 values $n^s = \exp\left(s\ln(n)\right)$ and interpretation in UD1 is via series approximation (appendix 1).

With such prescribed entities we have interpretation to any nominated degree of accuracy in UD1.

Interestingly, we note in passing that the relationship between ζ and the analytic prime numbers contains little direct information about analytic prime numbers.

Indeed, to establish that

$$\zeta(s) = \prod_{p\ \mathrm{prime}} (1-\frac{1}{p^s})^{-1} \text{ for } \sigma > 1$$

we only need (assuming uniqueness of factorisation)

$$\lim_{n\to\infty}\{\frac{1}{N^\sigma}+\frac{1}{(N+1)^\sigma}+\cdots\} = 0, \text{(Titchmarsh [2] page 1\&2)},$$

and this convergence requirement is quite weak.

Indeed, in UD2 for σ>1, with help from the ancients

$$1+\left(\frac{1}{2^\sigma}+\frac{1}{3^\sigma}\right)+\left(\frac{1}{4^\sigma}+\frac{1}{5^\sigma}+\frac{1}{6^\sigma}+\frac{1}{7^\sigma}\right)+\left(\frac{1}{8^\sigma}+\frac{1}{9^\sigma}+\frac{1}{10^\sigma}+\frac{1}{11^\sigma}+\frac{1}{12^\sigma}+\frac{1}{13^\sigma}+\frac{1}{14^\sigma}+\frac{1}{15^\sigma}\right)+\cdots$$

$$< 1+\frac{1}{2^{\sigma-1}}+(\frac{1}{2^{\sigma-1}})^2++(\frac{1}{2^{\sigma-1}})^3+.. = \frac{2^{\sigma-1}}{2^{\sigma-1}-1}.$$

So for N=1,2,3 ……. we may get a suitable estimate

$$\left| \sum_{1\le n\le N} \frac{1}{n^s} - \prod_{\substack{p<N \\ \text{p prime}}} \left(1-\frac{1}{p^s}\right)^{-1} \right|$$

via the uniqueness of factorisation of prime numbers and this is clearly an inductive argument. We cannot carry this inductive argument to the limit case for primes in rational arithmetic as we will see the ordering of all the prime structures assumes a domain of knowledge which includes the natural logarithm.

However, we are able to justify the 'analytic prime' interpretation of the original equation and we will not find numerical inconsistency between purely arithmetic endeavours and numerical analytic results.

Continuing with the arithmetic/analytic distinction, the arithmetic Möbius function μ_1 may be defined stepwise by

$$\mu_1(1) = 1, \qquad \text{for } N > 1, \qquad \text{if} \sum_{n\le N-1} \mu_1(n) \left[\frac{N}{n}\right] = 0 \text{ then } \mu_1(N) = 1, \qquad \text{else if}$$
$$\sum_{n\le N-1} \mu_1(n) \left[\frac{N}{n}\right] = 1 \text{ then } \mu_1(N) = 0, \qquad \text{else } \mu_1(N) = -1;$$

and the analytic Möbius function, μ_2 is defined by

$$\sum_{n\ge1} \frac{\mu_2(n)}{n^s}\zeta(s) = 1 \ (\sigma > 1).$$

For inductive numbers n we have $\mu_1(n) = \mu_2(n)$ and that is the extent of the relationship in UD1.

This relationship is somewhat like the limited capacity of rational arithmetic to describe irrational numbers. We may also view the analytic μ as a continuation of the arithmetic μ.

There is a very straight forward way of understanding the arithmetic/analytic distinction from the history of elementary number theory. If we count the number of numbers k with $1\le k\le N$ divisible by a prime number we have the relationship

$$N - 1 = \sum_{\substack{p\le N \\ \text{p prime}}} \left[\frac{N}{p}\right] - \sum_{\substack{p,q\le N \\ \text{p,q prime} \\ \text{p,q distinct}}} \left[\frac{N}{pq}\right] + \sum_{\substack{p,q,r\le N \\ \text{p,q,r prime} \\ \text{p,q,r distinct}}} \left[\frac{N}{pqr}\right] - \cdots$$

where distinct prime structure is included exactly once in the denominators. This counting does assume that numbers have an essentially unique factorisation in terms of prime numbers.

Thus, all that the equations

$$\sum_{n\le N} \mu(n)\left[\frac{N}{n}\right] = 1, \qquad (N = 1,2,3 \ldots\ldots)$$

are saying, assuming uniqueness of factorisation, is that every number greater than unity is divisible by a prime. It is thus not surprising that if we solve this system of equations for M(N) in UD1 we end up with M(N)=M(N) (see Braun [4] Appendix 1). Yet we see shortly the limit collection of equations contains enough information to derive the prime structure of the natural numbers and this puts us outside rational arithmetic.

Section 4

We now move to the drivers of RH unprovability in rational arithmetic.

The important logical chain is

$\{\boldsymbol{\mu}(\mathbf{n}): \mathbf{n} \geq \mathbf{1}\} \rightarrow \{\boldsymbol{\pi}(\mathbf{n}): \mathbf{n} \geq \mathbf{1}\} \rightarrow \{\mathbf{ln}(\mathbf{n}) : \mathbf{n} \geq \mathbf{1}\}$,

where the first implication (left to right) uses the arithmetic values in UD1 and the overall collection is on the 'limit' sets in UD2.

The essential idea is that limit type questions involving the μ values in a non trivial way necessarily require UD2 for resolution because the ordered values of μ lie behind the irrational logarithm values and are thus beyond mathematical inductive pattern. We cannot resolve questions which have unbounded distinct pattern by inductive argument in arithmetic.

For σ=Re{s}>1

$$\ln(\zeta(s)) = \sum_{k\geq 1} \frac{1}{k} p(ks) \text{ where } p(ks) = \sum_{p \text{ prime}} \frac{1}{p^{ks}}$$

and

$$\zeta(s) = \exp(\ln(\zeta(s)) = \exp(\sum_{k\geq 1} \frac{1}{k} p(ks)) = \prod_{k\geq 1} \exp(\frac{1}{k} p(ks))$$

We expand the RHS using the power series expansion for exp and group terms of the form

$p^{a_1}(b_1 s) p^{a_2}(b_2 s) \ldots\ldots p^{a_r}(b_r s)$ where $\sum a_i b_i = K$ (constant) for $K = 1,2, \ldots.$ and $r \geq 1$.

The expansion starts

$$\zeta(s) = 1 + p(s) + \left\{\frac{1}{2} p^2(s) + \frac{1}{2} p(2s)\right\} + \left\{\frac{1}{6} p^3(s) + \frac{1}{2} p(s) p(2s) + \frac{1}{3} p(3s)\right\} +$$

$$+ \left\{\frac{1}{24} p^4(s) + \frac{1}{4} p^2(s) p(2s) + \frac{1}{3} p(s) p(3s) + \frac{1}{8} p^2(2s) + \frac{1}{4} p(4s)\right\} + \cdots \quad \ldots (1).$$

We note in passing that in this grouping

the isolated coefficient of $p^a(bs) = \dfrac{1}{a!\, b^a}$.

The coefficients of other composite products are formed multiplicatively from these values.

We rearrange (1) as

$$p(s) = \zeta(s) - 1 - \left\{\frac{1}{2}p^2(s) + \frac{1}{2}p(2s)\right\} - \left\{\frac{1}{6}p^3(s) + \frac{1}{2}p(s)p(2s) + \frac{1}{3}p(3s)\right\} +$$

$$-\left\{\frac{1}{24}p^4(s) + \frac{1}{4}p^2(s)p(2s) + \frac{1}{3}p(s)p(3s) + \frac{1}{8}p^2(2s) + \frac{1}{4}p(4s)\right\} - \cdots \quad \ldots (2).$$

Equations (1) and (2) are sums and products of Dirichlet series and the equations may be expressed in terms of partial coefficient sums of such series (See appendix 1).

We may write out equation (2) in terms of the prime number counting function:

$$P_1(x) = [x] - 1 - \{\frac{1}{2}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_2\,(x) + \frac{1}{2}P_1(x^{\frac{1}{2}})\} - \{\frac{1}{6}P_3(x) + \frac{1}{2}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_1(\frac{x}{p^2}) + \frac{1}{3}P_1\left(x^{\frac{1}{3}}\right)\} +$$

$$-\{\frac{1}{24}P_4(x) + \frac{1}{4}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_2(\frac{x}{p^2}) + \frac{1}{3}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_1(\frac{x}{p^3}) + \frac{1}{8}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_1(\sqrt{\frac{x}{p^2}}) + \frac{1}{4}\sum_{\substack{p\geq 2\\ \text{p prime}}} P_1(x^{\frac{1}{4}})\}\ \ldots \quad (3)\,,$$

where

$$P_1(x) = \pi(x) \text{ and } P_k(x) = \sum_{\text{p prime}} P_{k-1}(\frac{x}{p}).$$

For the first few terms shown above this notation is adequate but for higher terms a more complex description is needed for the equation. However much the complexity of terms on the RHS of (3) they may be evaluated from prior values of the prime number sum function (see Appendix 2).

In the specific case (3) involving all the primes, the balancing term [x] produces an all important inductive link via [x]→[x]+1which allows the values of prime numbers to be evaluated stepwise. The identity yields a capacity to calculate the order and values of primes in arithmetic.

For example, from (3) with $P_1(1) = 0$, we have $P_1(2) = 1$, $P_1(3) = 2$, $P_1(4) = 2$, and so on. .

$P_1(x)$ not only counts primes in arithmetic, it allows the value of primes to be determined inductively in the broad sense (in principle) using just field addition and multiplication – an algebraic/arithmetic alternative to the sieve of Eratosthenes. $P_1(x)$ defines itself in (3) in a complicated recursive fashion.

We show that (3) and a corresponding equation involving M(x) leads quickly to an explanation of the Riemann hypothesis.

We show that the π values and the M values encode the same information in arithmetic and in the analytic realm.

Indeed, corresponding to (1) using a similar argument for the derivation, we have

$$\frac{1}{\zeta(s)} = 1 - p(s) + \left\{\frac{1}{2}p^2(s) - \frac{1}{2}p(2s)\right\} - \left\{\frac{1}{6}p^3(s) - \frac{1}{2}p(s)p(2s) + \frac{1}{3}p(3s)\right\} +$$

$$+ \left\{\frac{1}{24}p^4(s) - \frac{1}{4}p^2(s)p(2s) + \frac{1}{3}p(s)p(3s) + \frac{1}{8}p^2(2s) - \frac{1}{4}p(4s)\right\} + \cdots \quad \ldots(4)$$

which then yields

$$M(x) = 1 - P_1(x) + \{\frac{1}{2}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_2(x) - \frac{1}{2}P_1(x^{\frac{1}{2}})\} - \{\frac{1}{6}P_3(x) - \frac{1}{2}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\frac{x}{p^2}) + \frac{1}{3}P_1\left(x^{\frac{1}{3}}\right)\} +$$

$$+\{\frac{1}{24}P_4(x) - \frac{1}{4}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_2(\frac{x}{p^2}) + \frac{1}{3}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\frac{x}{p^3}) + \frac{1}{8}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\sqrt{\frac{x}{p^2}}) - \frac{1}{4}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(x^{\frac{1}{4}})\} \ldots \quad (5)\ .$$

We noted in [1] that not only is M step-wise determined in arithmetic from P_1 using (5) but if we rearrange (5) as

$$P_1(x) = 1 - M(x) + \{\frac{1}{2}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_2(x) - \frac{1}{2}P_1(x^{\frac{1}{2}})\} - \{\frac{1}{6}P_3(x) - \frac{1}{2}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\frac{x}{p^2}) + \frac{1}{3}P_1\left(x^{\frac{1}{3}}\right)\} +$$

$$+\{\frac{1}{24}P_4(x) - \frac{1}{4}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_2(\frac{x}{p^2}) + \frac{1}{3}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\frac{x}{p^3}) + \frac{1}{8}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(\sqrt{\frac{x}{p^2}}) - \frac{1}{4}\sum_{\substack{p\geq 2\\ p\text{ prime}}} P_1(x^{\frac{1}{4}})\} \ldots \quad (6)\ ,$$

then the values of $M(x)$ may be used to calculate the values of the prime number counting function in a step-wise fashion.

Clearly, this is not a sensible way of calculating things but the theoretical significance is far reaching.

We note that the same conclusions may be reached with a focus on $\ln(\zeta(s))$ rather than $p(s)$ which makes the expressions for the partial coefficient sums more straightforward but we would not be dealing quite so directly with $\pi(x)$.

Section 5

Separating rational arithmetic and analytic arithmetic

A number theorist restricting themself to rational arithmetic is undeniably allowed to think about how big $M(x)$ gets from time to time or how small $g(x)$ gets considering large values of x and these considerations can be tightened into what most would agree look like arithmetic propositions. If we look more closely at the evolution of why we would ask such questions we realise we are borrowing the limit concept from the development of the real and complex number systems.
We do have some examples of convergence in arithmetic. For example

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

This example belongs to the family of power series with rational coefficients which reduce to a ratio of two polynomials each with rational coefficients. i.e. there is an inductive pattern in the coefficients which reduces the series to the ratio of a couple of finite series which are rational for rational argument (singularities excluded).

We need to understand that from an arithmetic point of view if we contemplate (for example)

$$\sum_{n\geq 1}\frac{\mu(n)}{n} = 0 \quad (g(\infty) = 0).$$

there is no reduction or any kind upon which to latch an inductive argument in arithmetic for convergence.

This is not obvious at first, even though intuitively it is more likely than not, but we cannot readily discount the logical possibility that there is some inductive pattern in the μ values which somehow produce a finite proof of the above equation in arithmetic. The exclusion of this type of possibility opens up a way to explaining RH.
The conditional convergence $g(\infty)=0$ involves all the ordered prime factorisations of the natural numbers to determine the value of μ(n) for each n in order to consider the limit value. It is a question involving the analytic primes as discussed in section 2. This puts the inquiry squarely in UD2. The framing of the question of convergence immediately puts the inquiry into UD2. i.e. questions involving unbounded ordered distinct prime structure are questions about the analytic primes and the order of the analytic primes define the logarithm. To contemplate such problems resolvable in arithmetic is no more than folly.

The natural logarithm is the undisputed necessary path needed in order to gain access to non-trivial structural information involving the order of prime products. The logarithm is the can opener which opens up the multiplicative structure of the multiplicative natural number can.
It allows a small peep hole through which to access order of prime structure in a theoretical way.
We have shown that the arithmetic μ values imply the π values and vice versa in arithmetic. In the limit then either set of values enables the construction of ln(n) for the natural numbers n from arithmetic.

Indeed, as mentioned above, in the rational field we have for each prime $p\neq q$ (q fixed prime, $q=2$ for example),

$q^{\alpha(n_p)} < p^n < q^{\alpha(n_p)+1}$ and the sequence $\{\frac{\alpha(n_p)}{n}\}$ is Cauchy convergent $(=\frac{\ln(p)}{\ln(q)})$.

We thus see that since the ordering of the primes and prime powers is sufficient to order the natural numbers according to their prime factoring, we have the exact value of n via the coding $n=\exp(\ln(n))$. In short 'knowing' the ordering of all the primes and prime powers takes us to the natural logarithm on natural numbers and thus necessarily outside of UD1.

We have seen in Section 4 that the ordered μ values on the natural numbers lie the behind ordering of numbers by prime structure and hence in the limit case lie behind the natural logarithm on natural numbers. As the logarithm is beyond the reach of arithmetic so too are the ordered values of μ. i.e. as we cannot unlock all the ordered μ values in arithmetic we do not have enough to work with in arithmetic to establish $g(\infty)=0$.

Similarly, $M(x)=o(x)$ as $x\rightarrow\infty$ is unprovable in arithmetic.

i. e. $\lim_{x\rightarrow\infty}\frac{M(x)}{x}$ is a question about analytic primes.

Consequently, the order of $M(x)$ as $x\rightarrow\infty$ in the range [1/2, 1] is undecidable in UD1.
we cannot frame the question without being in UD2.

Yet on the computational side, aided by 0-1 type computer calculations, we remain in rational arithmetic in finding that sequential zeros of ζ in the critical strip. An exceptional zero found by computation would provide an arithmetic proof that RH was false. It would allow a narrowing of the range for the order estimate of $M(x)$ as $x\rightarrow\infty$.

Similarly, the proposition $M(x)=O(\sqrt{x})$ as $x\rightarrow\infty$ is undecidable in arithmetic. A computed multiple zero on $\sigma=1/2$ would prove in arithmetic that $M(x)\neq O(\sqrt{x})$ as $x\rightarrow\infty$ and hence all computed zeros will be simple zeros. Ironically, the weakness of rational arithmetic proves to be remarkably strong.

Appendix 1

We have noted in various discussions that the connection of $\zeta(s)$ to arithmetic in terms of known exact numerical values is very limited if not non-existent and in numerical work interpretation in UD1 is thus through numerical approximation.

Common prescribed functions in the theory which assist in the numerical interpretation are listed below:-

$$\exp(s) = 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots$$

$$\cos(s) = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \cdots$$

$$\sin(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \cdots$$

$$\ln(n) = \left(1 - \frac{1}{n}\right) + \frac{1}{2}\left(1 - \frac{1}{n}\right)^2 + \cdots$$

$$n^{\sigma+it} = \exp(\ln(n^\sigma))(\cos(t\ln(n) + i\sin\big(t\ln(n)\big)$$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad .$$

And for interpretation in $\sigma>0$,

$$\zeta(s) = (1 - \frac{2}{2^s})^{-1}(1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \quad) \ .$$

Appendix 1

Multiplication of Dirichlet series

Let

$$f_1(s) = \sum_{n\geq 1} \frac{a_1(n)}{n^s}, f_2(s) = \sum_{n\geq 1} \frac{a_2(n)}{n^s}, \dots, f_k(s) = \sum_{n\geq 1} \frac{a_k(n)}{n^s}, \qquad g(s) = \sum_{n\geq 1} \frac{g(n)}{n^s} \text{ and } h(s) = \sum_{n\geq 1} \frac{h(n)}{n^s}.$$

Let

$$A_1(x) = \sum_{n\leq x} a_1(n), \ \dots \quad A_k(x) = \sum_{n\leq x} a_k(n), \quad G(x) = \sum_{n\leq x} g(n) \text{ and }, \quad H(x) = \sum_{n\leq x} h(n).$$

If

$$f_1(s)f_2(s)..f_k(s)g(s) = h(s)$$

then

$$H(x) = \sum_{n_1\leq x} a_1(n_1) \sum_{n_2\leq x} a_2(n_2) \dots \sum_{n_k\leq x} a_k(n_k)\, G\left(\left[\frac{x}{n_1 n_2..n_k}\right]\right).$$

Clearly, the role of $g(s)$ and any of the $f_i(s)$may be interchanged.

In the case of $p^{k+1}(bs)$ the coefficient sum up to $[x]$ ($\mathrm{Cosum}_x(p^{k+1}(bs))$ is given by

$$\mathrm{Cosum}_x(p^{k+1}(bs)) = \sum_{n_1\leq x} l(n_1) \sum_{n_2\leq x} l(n_2) \dots \sum_{n_k\leq x} l(n_k)\, \pi\left(\left[\left(\frac{x}{n_1 n_2..n_k}\right)^{\frac{1}{b}}\right]\right)$$

where $l(n)=1$ if n is prime otherwise $l(n)=0$.

The values of composite coefficient sums derived from series with more than one primary component $p^{a_1}(b_1 s)p^{a_2}(b_2 s) \dots . p^{a_r}(b_r s)$ are implied via these expressions.

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