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The Riemann hypothesis undecidable in arithmetic (i)

by

Dr Peter Braun

Introduction

The upshot of this is that for practical purposes the Riemann hypothesis and the simplicity of the zeros will never be contradicted by numerical calculation.

One step up at a deeper theoretical logical level the talk about the smoothness of distribution of the prime numbers turns out to be a red herring – a product of the passion number theory has for asymptotic estimates- the distribution of primes couldn't be any worse or any better as we see we cannot think about all the primes in one go – even with the help of the Riemann zeta function.

Pure mathematics as taught is comfortable mapping one system into another preserving the shape of the initial system in terms of binary operations and ordering and maintaining a common base logic.

However, as we have discussed Braun [1], in mapping the rational numbers into the real numbers we require more in the way of logical assumptions at the theoretical level as we go beyond the inductive reach of arithmetic dealing with properties of uncountable entities.

We see shortly a true result using the full power of analysis, which has an interpretation in arithmetic, may be true or undecidable in arithmetic.

This is where intuitionism, and finiteism may come into play. Undecidability in arithmetic is seen as something just as natural as the natural numbers.

Analytic results which have interpretation in arithmetic but no proof in arithmetic.

Arithmetic has two well known states:-

The historical state where irrational numbers simply do not exist and the natural state where they are imbedded in the real numbers preserving the binary operations and the ordering.

Examination of the constructions involved in defining these two systems reveals that they may be considered as separate axiomatic systems –albeit closely related.

In relationship to each other, arithmetic and complex analysis have the property that in practical theory no contradiction of a numeric nature is possible between the two systems.

The term 'practical theory' is used here to imply that all sequences and series are prescribed or defined inductively via rational numbers. For example, the analytical theory of the Riemann zeta function deals with series which are essentially defined inductively.

It is practical theory in the sense that we are dealing with entities which take values and those values may be pointed to by rational approximations – an inductive and arithmetic activity.

This distinction is made relative to a higher conceptual level where general sequences and series are discussed, relating to some external properties which they do or do not have - for example, Cauchy convergence.

Some consider the natural numbers of the rational numbers and the natural numbers of the real numbers to be essentially the same natural numbers.

This is a safe attitude in that essentially numerical results gained by analytical methods are covered by inductive arguments from the base arithmetic but context is important. We may use the inductive arguments of arithmetic on prescribed series to prove there is no contradiction to be found in arithmetic derived from complex analysis, but the strength of such results in arithmetic is only one of true or undecidable if we disavow the existence of irrational numbers and remain in the deductive system of arithmetic. We need to be careful what we allow into our universe of enlightenment when we are talking about properties of 'natural numbers'.

We show that RH is undecidable in arithmetic and that RH is without contradiction in the analytic realm.

Numerical analysis involving estimations for the values of complex functions defined through prescribed (inductively defined) series is an activity in arithmetic.

A clue which points to the solution of the problem of RH is that Riemann was able to sit down and calculate the real values of the first few complex zeros of the zeta function. In estimating these zeros he was essentially using the inductive arguments of arithmetic without the higher assumptions of real and complex analysis. The imaginary parts of the zeros were approximations, but in principle, their values could be refined to higher degrees of accuracy. This activity is quite clearly inductive and arithmetical. Indeed, we have results that so many million consecutive zeros lie on $\sigma=1/2$ using numerical analysis and inductive arguments from the practical theory of the Riemann zeta function.

That the Riemann hypothesis (RH) may in some sense be undecidable does not have much openly reported discussion or literature from mainstream number theorists.

Let $\Theta = \text{lub}\{ \theta: \zeta(\sigma+it)\neq 0 \text{ for } \sigma>\theta\}$. We know that $1/2\leq\Theta\leq 1$.

Titchmarsh [2] shows caution when he writes 'It will be seen that a perfectly coherent theory can be constructed on this basis (Riemann Hypothesis), which perhaps gives some support to the view that the hypothesis is true'.

A corresponding observation applies for any Θ in the range $1/2\leq\Theta\leq 1$.

We propose here an argument with a relatively uncomplicated structure:

(i) Two axiomatic systems UD1 (part of inductive arithmetic Q) and UD2 (part of complex analysis)

(ii) No numerical result in UD1 will find contradiction in UD2 and no numerical result in UD2 will find contradiction in UD1

(iii) In UD1, no assumed rational value of Θ in [1/2, 1] leads to contradiction

(iv) All computed zeros of ζ in the critical strip lie on $\sigma=1/2$

(v) A corollary implies all such computed zeros will be simple.

The precise composition of UD1 and UD2 is not so important as the key immutable truth that no irrational number exists in UD1.

If we have a proof in UD1 of such and such there cannot be any content in the proof implying the existence of an irrational number. A purported proof in UD1 which necessarily violates this is either false in UD1 or unprovable in UD1. We cannot undo our knowledge of UD2 but we are able to hold that knowledge at arms length in UD1.

A neglect of this simple definition obscures an explanation of RH.

Working definitions for UD1 and UD2

UD1 - The inductive arithmetic of the rational field Q including numerical work involving series approximations to analytic functions defined inductively via prescribed power series.

UD2 – All the acceptable complex analysis required to develop the theory of the Riemann zeta function.

We can of course informally use power series willy nilly in arithmetic and go ahead in arithmetic getting approximations for values like $\zeta(2)$. We notice a numerical trend in this particular approximation activity but that is all.

We know from such things as the Skewes number that theory sometimes trumps numerical trends.

The rational status of $\zeta(2)$ in UD1 is undecidable in UD1 because it necessarily involves the existence of an irrational number – namely $\zeta(2)$.

Then if we have an arithmetic function $\theta(x)$ and a hypothetical asymptotic estimate

$$\theta(x) = x + \zeta(2)x^{1/2} + O(1) \text{ as } x \to \infty$$

the estimate is unprovable/undecidable in UD1.

In UD2 the value $\zeta(2)$ is exact but in UD1 if we focus on the truncated series it is the outcome of an activity (discovering a numerical trend) but not going beyond this.

The case we will be dealing with is quite closely related to this hypothetical example with the minor variation that the rational status of γ (Euler's constant) is undecidable in UD1 not because we know its rational status but because we need to venture into UD2 to pose the question about its status.

For example, for the divisor function $d(n)$, we have the well known analytic result

$$\sum_{n\le x} d(n) = x\ln(x) + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right) \text{ as } x \to \infty.$$

If we set $l(x) = \sum_{n\le x} \frac{1}{n}$, a corresponding analytic result is

$$\sum_{n\le x} d(n) = xl(x) + (\gamma - 1)x + O\left(x^{\frac{1}{2}}\right) \text{ as } x \to \infty.$$

Ignoring any possible difficulties with the error term, the constant term Euler's constant γ on the RHS makes the RHS inextricably linked to the analytic universe. Indeed, to ponder γ rational and provable in UD1 inevitably leads to circular argument. We cannot define γ exactly in UD1 without explicitly or implicitly referencing the natural logarithm and this function is not defined in UD1 except the trivial ln(1)=0.

Thus in UD1,

$$\sum_{n\le x} d(n) = xl(x) + O(x^1) \text{ as } x \to \infty.$$

is the best possible 'power' type asymptotic result.

We reintroduce the following notation from Braun [1] and the well known result:

$$\varphi(x) = \sum_{\substack{p\le x \\ p \text{ prime}}} \ln(p) = x + |O\left(x^{\Theta}\right) \text{ as } x \to \infty$$

where the symbol | means Θ is the least upper bound of numbers Δ such that

$$\sum_{\substack{p\le x \\ p \text{ prime}}} \ln(p) = x + O(x^{\Delta}) \text{ as } x \to \infty.$$

We know that Θ is the least upper bound of numbers Δ such that $\zeta(s)$ is zero free for $>\Delta$ where

as usual $s=\sigma+it$ and the assumption $1/2\le\Theta\le1$ cannot be contradicted in arithmetic.

We also know from the analytic theory with

$$M(x) = \sum_{n\le x} \mu(n).$$

where μ denotes the Möbius function that

$$M(x) = |O\left(x^{\Theta}\right) \text{as } x \to \infty.$$

This latter relationship is important because it establishes a direct relationship between the placement of the zeros of ζ and the asymptotic growth rate of a purely arithmetic function.

Now we also have

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} l(p) = [x] - \gamma\pi(x) + |O(x^{\Theta}) \text{ as } x \to \infty,$$

where π(x) is the prime number counting function.

The best possible order estimate in UD1 is

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} l(p) = O(x^1) \text{ as } x \to \infty.$$

We cannot get around γ to establish the order of the remainder term.

We do not need to think about the value of γ, it is cannot be defined in UD1.

In other words an order estimate of the form

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} l(p) = O(x^{\Delta}) \text{ as } x \to \infty$$

with any $\Delta: 1/2 \leq \Delta < 1$ is undecidable in arithmetic.

The possible result

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} l(p) = |O(x^1) \text{ as } x \to \infty$$

is then also unprovable in UD1.

Thus in UD1 we may choose any rational value for Θ in [1/2, 1] without contradiction.

By similar argument and the basic theory of ζ regarding the Möbius sum function M(x) it follows that $M(x^2) = O(x)$ as $x \to \infty$, cannot be contradicted in arithmetic.

Hence, all computed zeros in the critical strip will be simple zeros.

Notes

We see the undecidability discussed here is quite concrete in nature.

In simple terms, numerical analysis is used as an interface between arithmetic and analysis.

Arithmetic is the base where indisputable true theorems are put together. The construction of the real and complex numbers brings in a level of doubt because our arithmetic conversations can only ever include a countable number of entities. However, we are able to use the inductive strength of arithmetic via numerical analysis to be satisfied that the single analytical system of complex variable involving prescribed series cannot produce numerical contradiction.

Thus the search for a higher abstract theory in which to give an 'analytic' proof of RH in the philosophical sense would necessitate assumptions outside of those required for complex analysis.

References

[1] Braun P. B. Euler's constant and the Riemann hypothesis. www.peterbraun.com.au

[2] Titchmarsh E. C. The Theory of the Riemann Zeta Function. Oxford Scientific Publications. 1988.