The Riemann hypothesis is undecidable in arithmetic(v)
by
Dr Peter Braun

Notation and usage

Arithmetic and PA each denote finite rational Peano arithmetic, CV denotes the complex variable theory we need in modelling arithmetic primes via the theory of the ζ. RH denotes the Riemann's hypothesis that all zeros of ζ in the critical strip lie on $\mathrm{Re}\{s\} = 1/2$.
We use the restricted Landau notation $O(N^{\Delta})$, $o(N^{\Delta})$, $\Omega(N^{\Delta})$ and $\Omega_{\pm}(N^{\Delta})$ and within the context of the real or rational numbers, Δ is referred to as the order or asymptotic order of the estimate.
The appearance here of the exponent Δ involves $x^{\Delta} = \exp(\Delta(\ln(x))$ which is outside of PA.
When we use the above notation in PA with rational $\Delta=p/q$, we define for example
$M(N) = O\left(N^{p/q}\right)$ to mean $|M(N)|^{q} = O(N^{p})$ as $N \to \infty$.
In the context of this notation we omit 'as $N \to \infty$' as this condition is always the case.
When ε occurs it is an arbitrarily small number greater than zero and we may assume it is always rational.

The Mobius sum function of arithmetic

The Mobius function μ may be defined in elementary number theory on natural numbers by

$\mu(1) = 1;\ \mu(n) = (-1)^{r}$ if n is the exact product of r distinct prime numbers, else $\mu(n) = 0$.

In unique factorisation, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_r^{\alpha_r}$ $(n > 1)$ then since μ is weakly multiplicative

$$\sum_{g|n} \mu(g) = \prod_{1\le k\le r} \left(1 + \frac{\mu(p_k^1)}{p_k^1} + \frac{\mu(p_k^2)}{p_k^2} + \cdots \frac{\mu\left(p_k^{\alpha_k}\right)}{p_k^{\alpha_k}}\right) = 0.$$

Then, with [ ] denoting the greatest integer function, and counting the number of times $\mu(n)$ appears in the double summation,

$$1 = \sum_{1\le k\le N} \sum_{g|k} \mu(k) = \sum_{1\le n\le N} \mu(n)\left[\frac{N}{n}\right] \quad (N \ge 1).$$

The collective information these equations provide is that every number except 1 is divisible by a prime number. In other words, every number greater than 1 is either a prime or a composite. We cannot necessarily expect to derive too much information about prime numbers from this without further input, and, historically if not necessarily, this has come from real and complex analysis and the use of the logarithm.

Another definition for μ for N>1, following from the last equations is:- if

$$\sum_{1\le n<N} \mu(n)\left[\frac{N}{n}\right] = \begin{array}{lll} 2 & \text{then } \mu(N) = & -1, \\ 1 & \text{then } \mu(N) = & 0, \\ 0 & \text{then } \mu(N) = & 1. \end{array}$$

This latter definition provides a way of obtaining the value of μ(N) using all the preceding μ values thus hinting that the inductively derived μ appears to depend on an increasing amount of non-trivial decision making. The value of μ(N) depends on the prime factorisation of N and there is no way of expressing this value without in some way applying a definition of μ(N) either directly or indirectly and thus knowing some things about the prime number decomposition of N.
For example we need to know which numbers are primes prior to determining the value of μ on composite numbers.

We have a stepwise generation of {μ(1), μ(2), μ(3), …} with decision making through calculation at each step.

Unlike the open ended sequence {1,2,3,…} we appear to require an overall increasing amount of knowledge K(N) deriving the first N values in the μ sequence in the sense that K(N)→∞ as N→∞.
We might think of the μ values as 'non-inductive' or 'intelligent'. We discuss aspects of this in Braun ([1], [2], [3], [4] & [5]). In particular we note that the 'limit' μ values and the 'limit' π values may be derived from each other in PA and since the 'limit' π values define the ordering of the logarithms of the primes we would expect the arithmetic μ values to be such a series. We pick up this point in proposition 1 (corollary 2).

Numerical value of numbers

The value of a number is quite an abstract notion in the construction of the number systems, and value is defined initially in terms of the ordering of natural numbers and counting. The natural number b>a, if there exists k such that a+k = b. This simple definition follows through in the constructions of the integers, rational numbers and real numbers. We also note that in the ordering of the real numbers we are assuming an ability to distinguish between an un-countable number of entities, not something which can be established inductively. What we do inductively in arithmetic is define the notion of a rational Cauchy sequence which then leads on to the construction of the real numbers. In the real field the notion of value is somewhat tenuously associated with the unique place in the ordering.

For rational numbers when we want to give value a clear numerical definition, we make use of 'signed distance from the origin' as in d(a,0) = a, and with a>b, d(a, b) = d(a)-d(b). We may obtain a comparable d definition for irrational numbers α through the limit process by $d(\alpha, 0) = \lim_{n\to\infty} d(\alpha_n, 0) = \lim_{n\to\infty} \alpha_n = \alpha$ but in computation the limit values of an extended d function d(α,0) can only be approximated numerically for irrational α.
We want the irrational real numbers to have 'values' as an extension of this notion but we can only approximate these 'imaginary' values in PA to a degree of accuracy via the rational Cauchy sequences which define them. We define the real numbers and then overlay the ordering back on the rational numbers to support the imaginary notion of exact value on the irrational real numbers. The irrational numbers are viewed as place holders in the ordering having the

required properties in theoretical work with substance but not numerical value. In formulae the irrational numbers of particular interest have labels such as π, e, γ and so on. If they are not too interesting, we may use (CS) or some label constant like A to denote a rational Cauchy sequence with rational status unknown.

Proposition 1

$$\sum_{1\leq n\leq N} |\mu(n)| = \frac{6}{\pi^2}N + o(N) \text{ as } N \to \infty.$$

Proof

See for example Insuk K. and Cho M.H. [1]* and Braun [3].

* the internet draft version the writer viewed had some awkward typographical errors and omissions but overall the argument looked to be correct, and, in Braun [3] we note the similarity with a proof of the prime number theorem.

Note (added June 2020)

Tenenbaum [1] (page 46) uses elementary methods to obtain the stronger result

$$\sum_{n\leq N} |\mu(n)| = \frac{6}{\pi^2}N + O(\sqrt{N}) \text{ by deriving the formula } \sum_{n\leq N} |\mu(n)| = \sum_{n\leq\sqrt{N}} [\frac{\mu(n)}{n^2}].$$

Corollary 1

Let

$$M^+(N) = \sum_{\substack{n\leq N \\ \mu(n)>0}} \mu(n) \text{ and } M^-(N) = \sum_{\substack{n\leq N \\ \mu(n)<0}} \mu(n)\,.$$

Then

$$M^+(N) = \frac{3}{\pi^2}N + o(N) \quad \text{and } M^-(N) = \frac{-3}{\pi^2}N + o(N)\,.$$

Proof

This follows using the well -known analytic result $M(N) = M^+(N) + M^-(N) = o(N)$.

Corollary 2

$M(N) = o(N)$ and $Ng(N) = o(N)$ are both undecidable in PA.

Proof

In the interpretation of corollary 1, the irrational number $3/\pi^2$ is simply a place holder for this theoretical result, not altogether different from a region of painting in an artist's picture:- the

mathematical result is the art work and $3/\pi^2$ is one of the components which adds to the appeal of the end result. But $3/\pi^2$ does not have a numerical value: - it is an irrational number and exists at a different conceptual level. We are justified in saying $3/\pi^2$ exists because we have a non-trivial definition in a formal language and reasoning system but it does not have substance in a numerical sense. From the point of view of numerical measurement, the two terms on the RHS of the last equation are essentially incompatible.
$3/\pi^2$ is a place holder is the sense that we have $M^+(N) = (CS)N + o(N)$, where (CS) denotes a Cauchy sequence but the right hand side of the equation cannot express values in PA since (CS) is irrational.

Hence, each of the estimates

$$M^+(N) = \frac{3}{\pi^2}N + o(N) \text{ and } M^-(N) = \frac{-3}{\pi^2}N + o(N) \text{ is undecidable in PA.}$$

The sum of these two terms, $M(N)$, cannot be resolved in PA manipulation as the leading terms cannot be recognised in PA. Hence $M(N)=o(N)$ is undecidable.

The $Ng(N)= o(N)$ un-decidability result follows using Abel summation (end of proof).

From the early comments above about the M function being equivalent to the π counting function and this result, we conclude that the inductive sequence $\{\mu(1), \mu(2), \mu(3) \ldots\ldots\}$ does not provide the information in PA necessary for non-trivial o, O and Ω asymptotic estimates of the $\Delta>0$ variety.

Note
At this point, undecidable means it is futile to look for a proof bounded by PA but we know that both results are provable in real analysis. We might think it prudent to replace undecidable by unprovable in this context but the ongoing discussion will justify the choice of word here.

Modelling the arithmetic prime numbers in analytical number theory

The complex variable theory around the Riemann zeta function is a modelling of the finite but unbounded internal world of the arithmetic prime numbers. It is inductively based, tracing all the way back to equivalence classes of rational Cauchy sequences in the heartland of arithmetic.

With this orientation we use the subscript PA for arithmetic entities and the subscript CV for analytic entities. Clearly for N=1,2, 3 .....

$\mu_{PA}(N) = \mu_{CV}(N)$ in inductive arithmetic and likewise $M_{PA}(N) = M_{CV}(N)$.

In CV manipulations involving $\lim F\{M_{CV}(N)\}$ we do not have a meaning for $\lim \mu_{PA}(\infty)$ even though the results are correct in CV. e.g.

$$\frac{1}{\zeta(s)} = s\int_1^\infty \frac{M_{CV}(x)}{x^{s+1}}dx \quad (\mathrm{Re}\{s\} > 1).$$

We take is as read that if we have a result about rational numbers derived in complex variable theory which has numerical interpretation then we will not be able to produce specific numerical calculations which contradict the result. This is the best outcome we can hope for and has that certainty, simply due to the inductive nature of constructing the real and complex fields and the P or ~P strategy in progressing standard arguments. Arithmetic may also use indirect arguments but these are used in arithmetic without use of limiting processes. An exact result in CV which has numerical interpretation is simply a sophisticated generalisation of a result which cannot be contradicted numerically in PA.

Intuitively, because of the inductive nature of the roots of arithmetic we might expect a better matching of the complex analytic theory to problems of an additive nature in arithmetic compared to results and problems of a multiplicative nature, which may be less sympathetic to yielding answers in this realm. The stepwise uniqueness of factorisation of natural numbers is a result provable in arithmetic (Davenport [1]).

We have more direct conversation in modelling PA in that static universe generated by axioms, definitions, standard logic and a much stricter language compared to modelling aspects of the external world which has more movable parts, but the notion of matching or alignment is applicable in both cases.

In examining the application of the theory of the Riemann zeta function and the analytic primes compared to the arithmetic primes we find a point in the decision making where the ζ function is no longer relevant and we need to call on PA to resolve the proposition RH:- an important case where un-decidability in PA carries weight in CV. Sometimes undecidable in PA means unprovable in PA and sometimes it means undecidable full stop.

When we run into anomalies between PA and CV in numerical problems there is the necessity to reconcile the anomaly in relation to the P or ~P steering wheel, rather than rejecting outright some undecidable assumption leading to the anomaly. In the theory of the Riemann zeta function in modelling the analytic primes the 'cracks' in differences between PA and CV become apparent on the line Re{s}=1/2. The resolution of the dispute on this boundary leads to an unusual outcome for the Riemann hypothesis.

Background

The advance of a non-trivial Ω asymptotic order estimate for $M_{CV}(N)$ stems from the functional equation for ζ.

The deriving of the functional equation through an application of

$$\theta(1/x) = \sqrt{x}\,\theta(x)\,, \quad \text{where } \theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} \quad x > 0\,, \quad \text{is far removed from PA}\,.$$

With the place holders √x, π and e appearing in the θ identity, the ζ functional equation may as well be declared an extraordinary result in logic and language for PA to digest, yet accepted as pointwise reliable in numerical verification.

The existence of zeros on Re{s} =1/2 only has a tenuous connection with the arithmetic Möbius function and it is the functional equation and the Riemann Ξ function and finding the sign change in Ξ(t) which produces a zero on Re{s} = ½ and we only need one zero to get a non-trivial Δ type Ω result for $M_{CV}N$). The existence of a zero of ζ on Re{s}= ½ implies that $M_{CV}N) = \Omega\left(N^{\frac{1}{2}-\varepsilon}\right)$ since otherwise ζ is zero free on Re{s} = ½.

We cannot contradict this in PA and using the P or ~P steering wheel we dutifully align $M_{PA}(N) = \Omega\left(N^{\frac{1}{2}-\varepsilon}\right)$ in the modelling process. Not surprisingly, protracted numerical investigation of $M_{PA}(N)$ does not contradict this. The stronger result $M_{CV}(N) = \Omega_{\pm}\left(N^{\frac{1}{2}-\varepsilon}\right)$, is also provable in CV and is nicely exhibited in extensive numerical investigations as we would expect.

To improve the trivial estimate $M_{PA}(N) = \Omega(N^0)$ to $M_{CV}(N) = \Omega\left(N^{\frac{1}{2}-\varepsilon}\right)$ in the modelling, in one fell swoop, is a breathtaking achievement for the mathematics which produced the result not to mention the mathematician who found it. ζ appears in every way a very good imaginary friend.

Yet CV wants to say to PA that all our numerical results are true because we can't contradict them.

We next discuss why this Ω result is at the end of the usefulness of knowledge of ζ properties and we can only get closer and closer approximations to RH by numerical calculation.

A closing argument

Intuitively, in PA we might view the validity of the P or ~P steering wheel in CV, as a weaker argument than the strength of axiomatically controlled inductive arguments in PA and we should also keep in mind the view that CV is derived from PA in the sense of Kronecker's metaphor:-

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"
(roughly)
("God made the natural numbers, all else is the work of man").
The rational field PA in this case substitutes for the entity 'natural numbers' in the quotation.

The idea of unification through set theory is an approach with marvellous application in the external world and we may be comfortable with it having the same status in the internal world of prime number theory but it does not prevent us separating finite rational Peano arithmetic from the complex variable theory of the ζ function.

The CV result $M_{CV}(N) = \Omega(\sqrt{N})$ (Titchmarsh [1]) presents a problem for PA.

Proposition 2 (weak Riemann hypothesis)

The truth values of the propositions $M_{PA}(N) = \Omega(\sqrt{N})$ and $M_{PA}(N) = o(\sqrt{N})$ cannot be resolved.

Proof:

The result $M_{CV}(N) = \Omega(\sqrt{N})$ is independent of the modelling of primes via ζ theory.
Indeed, the sketch of the proof in Titchmarsh [1] that $M_{CV}(N) = \Omega(\sqrt{N})$ is true provided RH is one of true or false. Then in CV theory the proposition is an unconditional result independent of

the truth value of RH.
However, prior to that understanding we have the result that $M_{PA}(N) = \Omega(\sqrt{N})$ is undecidable and this implies that $M_{PA}(N) = o(\sqrt{N})$ is also undecidable and independent of ζ modelling.

The independence of these two estimates in CV theory of ζ, clears the way to invoke the undecidability that has been established in PA:- it cannot be over-ruled by if and but's in ζ theory.

We may thus choose the truth values of these two propositions without contradiction in PA in the two possible cases:-

First case:-
$t \equiv \left(M_{PA}(N) = \Omega(\sqrt{N})\right)$ leaves open either RH true or RH false
$f \equiv (M_{PA}(N) = o(\sqrt{N})$ leaves open either RH true or RH false

Second case:-

$t \equiv (M_{PA}(N) = o(\sqrt{N})$ means computed zeros of ζ on the critical strip lie on Re{s}=1/2 and are simple zeros
$f \equiv \left(M_{PA}(N) = \Omega(\sqrt{N})\right)$ leaves open either RH true or RH false

None of these four outcomes can be contradicted numerically in PA.

That most annoying of all questions:- 'what if there is an offline zero?', and we only need one, has to wait to be asked until we have finished calculating all the simple zeros on Re{s} =1/2.

Notes:

Arithmetic can turn the tables and argue that whilst RH is not true, it cannot be contradicted in computation and this is a most satisfactory reconciliation of the problem in CV. The extensive numerical evidence which in a sense, is neither here nor there, is awfully consistent with this argument.

One would expect that the existence of a product formula and a functional equation in each of a wider class of zeta functions would produce the same phenomena in terms of numerically calculated zeros provided there is a comparable 'irrational' sum function like $M_{PA}(N)$ from which to gain leverage.

The discussion here does not in any way disagree with results like that of Riele and Oyyzko or any known work in the more exotic theory trying to determine better Ω results for $M_{CV}(N)$ since they are independent of the truth of RH in limit type proofs. They are however, results about $M_{CV}(N)$ but not $M_{PA}(N)$. The ζ function modelling no longer has relevance where the exactness is required of **all** zeros of ζ in the critical strip lying on Re{s}=1/2.

The reader may find interest in (Weyl [1]) for an historical perspective.

References:-

[1] Davenport H.
The Higher Arithmetic. First Harper Torchbook edition published 1960. (Chapter 1)

[1] Insuk K. and Cho M.H.
On the bound for the absolute value of Möbius function. Honam Mathematical J. 36 (2014) No.2.)

[1] Ollyzko A.M. & te Riele H.J.J.
(1985) Disproof of Merten's Conjecture. Journal für de reine und angewandte Mathematik;357; 136-160.

[1] Tenenbaun G. Introduction to analytic and probabilistic number theory. Cambridge studies in advanced mathematics: 46. CUP (English edition 1995) Pages 167-168.

[1] Titchmarsh E. C. The Theory of the Riemann Zeta-function, 2nd edition
(Revised by D.R. Heath-Brown) Oxford Science Publications. Pages 371-372.

[1] Weyl. H. Consistency in Mathematics. Mathematical Lectures. Rice Institute May 22-23, 1929.

Web References (https://www.peterbraun.com.au)

[1] The Riemann hypothesis is undecidable in rational arithmetic (i)
[2] The Riemann hypothesis is undecidable in rational arithmetic (ii)
[3] The Riemann hypothesis is undecidable in rational arithmetic (iii)
[4] The Riemann hypothesis is undecidable in rational arithmetic (iv)
[5] Algebra of Numbers (ii).