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The Riemann hypothesis is undecidable in rational arithmetic (iv)
by
Dr Peter Braun

The formation of a strategy for proving the Riemann hypothesis (RH) using a proposition or a collection of propositions un-decidable in rational arithmetic has been discussed in Braun [1], [2]& [3] and in various other discussions on the website over the last ten years.
We clarify the argument in Braun [3] and derive a best possible arithmetic estimate for the count of prime numbers up to x.

Big O, Ω and little o terms will always be in the context of the argument tending to infinity.

UD1 will denote the axiomatic system of finite rational arithmetic and UD2 will denote the separate axiomatic system of complex analysis developed as a numerically consistent extension of UD1.

Let

$$l(x) = \sum_{1\le k\le x} \frac{1}{k}, \qquad \pi(x) = \sum_{\substack{p\le x \\ p \text{ prime}}} 1, \qquad li(x) = \int_2^x \frac{dt}{\ln(t)}, \qquad \varphi(x) = \sum_{k\ge 1} \sum_{\substack{p^k\le x \\ p \text{ prime}}} \frac{\ln(p)}{k},$$

and $M(x) = \sum_{n\le x} \mu(n)$ and $g(x) = \sum_{n\le x} \frac{\mu(n)}{n}$ where μ is the Möbius function.

ζ denotes the Riemann zeta function and the complex variable s is defined by s=σ+it.

Lemma 1

$$li(x) - \sum_{2\le n\le x} \frac{1}{\ln(n)} = O(1) \text{ and } \sum_{2\le n\le x} \frac{1}{\ln(n)} - \sum_{2\le n\le x} \frac{1}{2\left(l(n) - l(\sqrt{n})\right)} = O(\sqrt{x}).$$

Proof

The first estimate follows using the truncated Euler-Maclaurin summation formula.

Similarly, using the same summation formula:-

$$\ln(n) = l(n) - \gamma + O\left(\frac{1}{n}\right).$$

Thus

$$\frac{1}{2}\ln(n) = l(\sqrt{n}) - \gamma + O\left(\frac{1}{\sqrt{n}}\right).$$

$$\frac{1}{2}\ln(\mathrm{n}) = \mathrm{l}(\mathrm{n}) - \mathrm{l}(\sqrt{\mathrm{n}}) + \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{n}}}\right).$$

$$\ln(\mathrm{n}) = 2(\mathrm{l}(\mathrm{n}) - \mathrm{l}(\sqrt{\mathrm{n}})) + \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{n}}}\right).$$

Then

$$\sum_{2\le n\le x}\frac{1}{\ln(n)} - \sum_{2\le n\le x}\frac{1}{2\left(\mathrm{l}(n) - \mathrm{l}(\sqrt{n})\right)} = \mathrm{O}\Big(\sum_{2\le n\le x}\frac{1}{(\mathrm{l}^2(n))\sqrt{n}}\Big) = \mathrm{O}(\sqrt{x}) \ .$$

Proposition 1

$$\pi(x) = \sum_{2\le n\le x}\frac{1}{2(\mathrm{l}(n) - \mathrm{l}(\sqrt{n}))} + \ \mathrm{O}\left(\mathrm{l}(x)(x^{\frac{1}{2}})\right) \ \equiv \text{Riemann hypothesis.}$$

Proof

It follows immediately from the classical/modern analytical result ([Ingham [1]) page 83 that

$\mathrm{li}(x) - \pi(x) = \mathrm{O}\left(x^{\frac{1}{2}}\ln(x)\right) \equiv$ RH and from the estimates in Lemma 1.

Notes

The ‘simplicity’ of proposition1 must surely mean it is ‘well known’ but it is new to the author.

For completeness we show a line of argument which explains a connection between li(x) and π(x). An estimate for π(x) using an arithmetic counting function is closer to home than the somewhat analytical li(x). To this end we prove in some detail the sufficient condition in proposition 1.

Proposition 2

$$\pi(x) = \sum_{2\le n\le x}\frac{1}{2(\mathrm{l}(n) - \mathrm{l}(\sqrt{n}))} + \ \mathrm{O}\left(x^{\frac{1}{2}}\mathrm{l}(x)\right) \ \text{implies the Riemann hypothesis.}$$

Proof

For X =f, g, h …. we use the notation $X_a(s)$ to denote a function analytic for Re{s}>a.

Let

$$\theta(s) = \sum_{n\ge 2}\frac{1}{\ln(n)\, n^s} \ (\sigma = \mathrm{Re}\{s\} > 1), \qquad L(x) = \sum_{n\le x}\frac{1}{\ln(n)} \ \text{ and } \ \mathrm{pi}(x) = \sum_{2\le n\le x}\frac{1}{2(\mathrm{l}(n) - \mathrm{l}(\sqrt{n}))} \, .$$

We assume

$$\pi(x) - \mathrm{pi}(x) = \mathrm{O}(x^{\frac{1}{2}}\mathrm{l}(x))$$

Also recall that Lemma 1 establishes

$$L(x) - pi(x) = O(\sqrt{x}).$$

These two estimates will be applied to integrals below.

Clearly for σ>1

$$\theta'(s) = 1 - \zeta(s) \ldots\ldots (1).$$

We may assume the analytic continuation of θ in the complex plane.

On the other hand for σ>1

$$\ln \zeta(s) = \sum_{p\ \text{prime}} \frac{1}{p^s} + g_{\frac{1}{2}}(s) = s\int_2^x \frac{\pi(x)}{x^{s+1}} dx + g_{\frac{1}{2}}(s)$$

$$\ln \zeta(s) = s\int_2^x \frac{\pi(x) - pi(x)}{x^{s+1}} dx + s\int_2^x \frac{pi(x)}{x^{s+1}} dx + g_{\frac{1}{2}}(s)$$

$$\ln \zeta(s) = s\int_2^x \frac{\pi(x) - pi(x)}{x^{s+1}} dx + s\int_2^x \frac{pi(x) - L(x)}{x^{s+1}} dx + s\int_2^x \frac{L(x)}{x^{s+1}} dx + g_{\frac{1}{2}}(s)$$

The first integral defines a function which is analytic in Re{s} >1/2 (by assumption), the second integral defines a function which is analytic for Re{s}>1/2 (from lemma 1) and the third integral is θ(x) (by definition)

Thus in σ>1/2

$$\ln \zeta(s) = \theta(s) + u_{\frac{1}{2}}(s).$$

Consequently in σ>1/2

$$\frac{\zeta'(s)}{\zeta(s)} = \theta'(s) + v_{\frac{1}{2}}(s) \ldots\ldots\ldots\ldots (2)$$

Using (1) and (2) we thus have

$$\frac{\zeta'(s)}{\zeta(s)} + \zeta(s) = v_{\frac{1}{2}}(s).$$

Clearly then,

ζ is free from zeros in Re{s} >1/2.

We note this final equation underlies another modern/classical analytical result:-

$\text{Cosum}_x \frac{\zeta'(s)}{\zeta(s)} + \text{Cosum}_x \zeta(s) = O(\ln^2(x) x^{\frac{1}{2}}) \equiv \text{RH}$, where Cosum_x indicates the partial coefficient sum of the indicated Dirichlet series.

Proposition 3

$$\pi(x) = \sum_{2 \le n \le x} \frac{1}{2\left(l(n) - l(\sqrt{n})\right)} + O\left(l(x)x^{\frac{1}{2}}\right).$$

Proof

We have discussed in [3] why

$$M(x) = \sum_{n \le x} \mu(n) = o(X) \text{ and } g(x) = \sum_{n \le x} \frac{\mu(n)}{n} = o(1)$$

are both un-decidable in UD1.

Let Re{s}=Θ be the largest value such that ζ(s) ≠0 for Re{s}>Θ and RH(Θ) be the proposition that ζ(s) ≠0 for Re{s}>Θ.

In UD2

$M(x) = 0 + \Omega(x^{\Theta})$ [1] Titchmarsh pages 371 − 372 for the case $\Theta = \frac{1}{2}$

$\varphi(x) = x + \Omega(x^{\Theta})$ [1] Ingham pages 100 − 103

$\pi(x) = li(x) + \Omega((x^{\Theta-\varepsilon})$ [1]Ingham pages 103 − 104.

We can give arithmetic interpretation to the LHS of these estimates giving:-.

$$M(x) = 0 + \Omega(x^{\Theta})$$

$$\sum_{\substack{p \le x \\ p \text{ prime}}} l(p) + \pi(x) \sum_{n \le \sqrt{x}} \sum_{m \le \sqrt{x}} \frac{g\left(\frac{x}{nm}\right)}{nm} = x + \Omega(x^{\Theta-\varepsilon}) \quad \text{(using proposition 4)}$$

and

$$\pi(x) = \sum_{2 \le n \le x} \frac{1}{2(l(n) - l(\sqrt{n}))} + \Omega(x^{\Theta-\varepsilon}).$$

Another important basic connection involving these functions is their logical equivalence, with error term, to RH(Θ). i.e.

$$M(x) = 0 + O(x^{\Theta}) \equiv RH(\Theta) \quad \ldots\ldots(2)$$

$$\sum_{\substack{p \le x \\ p \text{ prime}}} l(p) + \pi(x) \sum_{n \le \sqrt{x}} \sum_{m \le \sqrt{x}} \frac{g\left(\frac{x}{nm}\right)}{nm} = x + O(x^{\Theta+\varepsilon}) \equiv RH(\Theta)$$

and

$$\pi(x) = \sum_{2 \le n \le x} \frac{1}{2(l(n) - l(\sqrt{n}))} + O(x^{\Theta+\varepsilon}) \equiv RH(\Theta).$$

The theory of ζ necessarily involves using the natural logarithm and the extension in complex numbers so these three equivalences involving estimates are necessarily theorems in UD2.

Since LHS of (2) is undecidable in arithmetic for $0 \leq \Theta < 1$ in arithmetic we may choose any value in this range whilst we are in rational arithmetic without finding numerical contradiction.

However the theory of ζ in UD2 requires $\Theta \geq 1/2$ and this then is a new requirement on UD1 that $\Theta \geq 1/2$. In UD2 we may ask the question:- what if we find the prediction of a zero in $\mathrm{Re}\{s\} > \Delta > 1/2$? This would seem to require at adjustment to the imposition on UD1 to $\Theta \geq \Delta > 1/2$ in much the same way as the earlier matching of $\Theta \geq 1/2$?

To reject this possibility we need to recall that the whole story of the Riemann zeta function has been made up from UD1 using the invention of the real and complex numbers and assumptions outside of UD1.

In UD1 the un-decidability of the range for the value of Θ for the $M(x)$ estimate remains intact.

Any choice of Θ in this range defines a complex analysis UD2(Θ) which cannot be contradicted in arithmetic. We thus see that there is not just one RH but an uncountable number RH(Θ) none of which can be contradicted in arithmetic calculation.

However, regardless of which RH(Θ) we choose there is only one calculation method, independent of Θ, for locating zeros in $0 \leq \mathrm{Im}\{s\} \leq T$. The choice $\Theta = 1/2$ calculation will always show zeros only on $\mathrm{Re}\{s\} = 1/2$.

We cannot find zeros off the line $\mathrm{Re}\{s\} = 1/2$ by calculation but from the assumptions of arithmetic we cannot contradict the existence of one in UD2(Θ) for $½ < \Theta \leq 1$.

There is no contradiction here because we are in different systems rather than different logical cases in one system.

Proposition (3) now follows from (4).

The formal RH is UD2 now follows noting that the assumed existence of a zero of ζ in $\sigma > 1/2$ could in principle be located by numerical approximation consistent with the theory in UD2.

Notes

(One) The appearance of $[\sqrt{n}]$ in the denominator of the terms defining pi(x) causes a problem in the limiting case in rational arithmetic.

The arithmetic condition $1 \leq k^2 \leq [x]$ requires translation to the analytic condition $1 \leq 2\ln(k) \leq \ln(x)$ to have meaning in the limiting case.

We see this sort of 'irrationality factor' is also imbedded in the definition of the Liouville function λ in the relationship

$$\sum_{n \leq x} \lambda(n)[\frac{x}{n}] = [\sqrt{x}].$$

Indeed both

$$\sum_{n\le x}\mu(n) = o(X) \text{ and } \sum_{n\le x}\frac{\mu(n)}{n} = o(1)$$

are un-decidable in rational arithmetic (website [3]).

Then since there is an arithmetic manipulation which shows the orders of $M(x)$ and the corresponding Liouville sum function $S(x)$ are the same we conclude that the order of $S(x)$ in un-decidable in arithmetic.

(Two) The strategy in the above discussion for deriving a workable way of estimating $\pi(x)$ in rational arithmetic is to side step the exact value of Euler's constant γ.

We argue in [1] that since γ is inextricably linked to the natural logarithm it becomes

an immutable block in rational arithmetic to proving or disproving an RH equivalent

$$\sum_{\substack{p\le x\\ p \text{ prime}}} \ln(p) = x + O(x^{\frac{1}{2}+\varepsilon}) \text{ in the form } \sum_{\substack{p\le x\\ p \text{ prime}}} l(p) - \gamma\pi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$$

We may use the symbol γ in writing up analytic results where it represents its exact value but we cannot represent its exact value in the writing of rational arithmetic. To substitute a suitable arithmetic approximation to γ runs into the problem of recognising the approximation in the first place.

Suppose we were able to prove

$$\sum_{\substack{p\le x\\ p \text{ prime}}} l(p) - CS\pi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$$

where CS denotes a Cauchy convergent sequence. We would only be in rational arithmetic if we could prove γ was rational in rational arithmetic and the logarithm connection with γ excludes this possibility.

We may still ask: what happens if we to side step γ in the statement:

$$\sum_{\substack{p\le x\\ p \text{ prime}}} l(p) - \gamma\pi(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \ ?$$

We are able to convert this into a purely arithmetic statement using Proposition 4:-

$$\sum_{\substack{p\le x\\ p \text{ prime}}} l(p) + \pi(x)\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}}\frac{g\left(\frac{x}{nm}\right)}{nm} = x + O(l(x)^2\sqrt{x}) \ .$$

The answer to the above question is then - we just end up with another arithmetic statement unprovable in rational arithmetic equivalent to the Riemann hypothesis.

Proposition 4

$$\gamma = -\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}} \frac{g\left(\frac{x}{nm}\right)}{nm} \quad + \quad O\left(\frac{1}{\sqrt{x}}\right) \text{ as } x \to \infty.$$

Proof

We derive this from a slightly more general case.

Let $f(s) = \sum_{n\ge1} \frac{a(n)}{n^s}$, $\mathrm{Cosum}_x f(s) = \sum_{n\le x} a(n)$, $g(s) = \sum_{n\ge1} \frac{b(n)}{n^s}$, $\mathrm{Cosum}_x g(s) = \sum_{n\le x} b(n)$,

$f(s)g(s) = 1$ and $f_{\sqrt{x}}(s) = \sum_{n\le\sqrt{x}} \frac{a(n)}{n^s}$.

The coefficients in the Dirichlet series

$$(f_{\sqrt{x}}(s)g(s) - 1)^2 = f^2_{\sqrt{x}}(s)g^2(s) - 2\, f_{\sqrt{x}}(s)g(s) + 1$$

are zero for 1≤n≤x.

Multiplying the RHS by the series1/g(s) we thus see that the coefficients of the Dirichlet series

$$f^2_{\sqrt{x}}(s)g(s) - 2\, f_{\sqrt{x}}(s) + f(s)$$

are zero for 1≤n≤x.

i. e. $\mathrm{Cosum}_n(f^2_{\sqrt{x}}(s)g(s)) - 2\mathrm{Cosum}_n\left(f_{\sqrt{x}}(s)\right) + \mathrm{Cosum}_n\big(f(s)\big) = 0$ for 1≤n≤x.

At n=[x] we thus have

$$\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}} a(n)a(m)B(\frac{x}{nm}) - 2\sum_{n\le\sqrt{x}} a(n) + \sum_{n\le x} a(n) = 0.$$

With the choice $a(n) = 1/n$ and $b(n) = \mu(n)/n$ we thus have

$$\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}} \frac{1}{nm} g(\frac{x}{nm}) = l(x) - 2l(\sqrt{x}) \text{ and using } l(x) = \ln(x) + \gamma + O(\frac{1}{x})$$

we obtain proposition 4.

Proposition 4 which is probably deeply imbedded in the notes of numerous people who have thought about this sort of thing does seem to have a rather large number of interesting features in thinking about the interface of arithmetic and prescribed analysis (UD1 and UD2).

As discussed, $\lim_{x\to\infty} \sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}} \frac{1}{nm} g(\frac{x}{nm})$ cannot be proven in the writing of rational arithmetic.

We noted in [3] that

$$\lim_{n\to\infty} g(n) = 0$$

although true, is unprovable in rational arithmetic, with the stumbling block being the irrational $\pi^2/6$.

We would expect the trivial estimate

$$\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}}\frac{O(1)}{nm} = O\big(l^2(n)\big) \text{ to be best possible}$$

in rational arithmetic.

Another choice a(n)=1, b(n) = μ(n) leads to

$$M(x) - 2M\big(\sqrt{x}\big) = -\sum_{n\le\sqrt{x}}\sum_{m\le\sqrt{x}}\mu(n)\mu(m)[\frac{x}{nm}]$$

where $M(x) = \sum_{n\le x}\mu(n)$.

This result is discussed [4] with the conclusions in [4] similarly noting the arithmetic condition of $1\le n^2\le x$ needs to be translated to $2\ln(n)\le\ln(x)$ in the limiting case $x\to\infty$.

References

[1] Ingham A.E. The distribution of prime numbers. Cambridge Tracts in Mathematics and Mathematical Physics. No. 30. The distribution of prime numbers. CUP 1995.

[1] Titchmarsh E. C. The Theory of the Riemann Zeta-function, 2nd edition (Revised by D.R. Heath-Brown) Oxford Science Publications.

Web References (www.peterbraun.com.au)

[1] The Riemann hypothesis is un-decidable in rational arithmetic (i)

[2 The Riemann hypothesis is un-decidable in rational arithmetic (ii)

[3] The Riemann hypothesis is un-decidable in rational arithmetic (iii)

[4] Another irrational sums argument for the Riemann hypothesis