Algebra of Number Forms (III)

Dr Peter Braun

Introduction:

Let $FLT(p) \equiv$ For $p > 2, X^p + Y^p + Z^p = 0$ in integers implies $XYZ = 0$.

We use a simple recurrence relationship mentioned in Algebra of Numbers (II) to argue that there are no 'stand alone' contradictions to FLT(p). That is: if an example were exhibited in arithmetic, it would belong to a parameter family of solutions $X(u, v..), Y(u, v \dots)$ and $Z(u, v \dots)$.

Let

$P_N = X^N + Y^N + Z^N, \;\; e_1 = X + Y + Z, \;\; e_2 = XY + YZ + ZX$ and $e_3 = XYZ$.

Then

$$X^N + Y^N + Z^N = (X^{N-1} + Y^{N-1} + Z^{N-1})(X + Y + Z) - \{X\left(Y^{N-1} + Z^{N-1}\right) + Y\left(X^{N-1} + Z^{N-1}\right) + Z\left(X^{N-1} + Y^{N-1}\right)\}$$

Hence

$$P_N = P_{N-1}P_1 - \{XYY^{N-2} + XZZ^{N-2} + XYX^{N-2} + YZZ^{N-2} + ZXX^{N-2} + ZYY^{N-2}\}$$

$$= P_{N-1}P_1 - \{XY\left(P_{N-2} - Z^{N-2}\right) + XZ\left(P_{N-2} - Y^{N-2}\right) + YZ\left(P_{N-2} - X^{N-2}\right)\}$$

$$= P_{N-1}P_1 - e_2P_{N-2} + e_3P_{N-3}\ .$$

If $P_1 = 0$ we see $e_2 = -\frac{1}{2}P_2$ and $e_3 = \frac{1}{3}P_3$, and hence with $P_1 = 0$,

$6P_N = 3P_2P_{N-2} + 2P_3P_{N-3}$.

An approach to FLT(p) in arithmetic (UD1)

$FLT(p) \equiv X^p + Y^p + Z^p = 0$ with $XYZ \neq 0$ has no solutions in integer for $p>2$.

Suppose for odd prime p, $\quad X^p + Y^p + Z^p = 0$ in coprime integers with $XYZ \neq 0$.

We see from above recurrence relationship that for all integers N:

$$(X^{Np} + Y^{Np} + Z^{Np}) = -((XY)^p + (YZ)^p + (ZX)^p)(X^{Np-2p} + Y^{Np-2p} + Z^{Np-2p}) + (XYZ)^p(X^{Np-3p} + Y^{Np-3p} + Z^{Np-3p}).$$

Then for each odd $N \geq 1$ with $XYZ \neq 0$, this may be written in integers as

$$\frac{(X^{Np} + Y^{Np} + Z^{Np})}{(XYZ)^p} = \frac{-((XY)^p + (YZ)^p + (ZX)^p)(X^{Np-2p} + Y^{Np-2p} + Z^{Np-2p})}{(XYZ)^p} + 2\left(\frac{X^{Np-3p} + Y^{Np-3p} + Z^{Np-3p}}{2}\right) \dots\dots (1).$$

We next show for all odd N such that (pXYZ, N(N-2)=1, the three terms in this integer equation are all coprime to XYZ.

Indeed, suppose (pXYZ, N(N-2)=1. Then

$$X^{Np} + Y^{Np} + Z^{Np} = ((X^p + Y^p + (-Y^p))^N + Y^{Np} + Z^{Np}.$$

$$= \left(X^p + Y^P\right)\{G_1(X, Y)\left(X^p + Y^P\right) + NY^{Np-p}\} + Z^{Np}.$$

i.e.

$$X^{Np} + Y^{Np} + Z^{Np} = Z^p\{G_1(X,Y)Z^p + NY^{Np-p}\} + Z^{Np}.$$

Thus $\dfrac{X^{Np} + Y^{Np} + Z^{Np}}{Z^p}$ is coprime to Z and so $\dfrac{X^{Np} + Y^{Np} + Z^{Np}}{(XYZ)^p}$ is coprime to XYZ.

With N odd and p an odd prime we have the corresponding properties as above with N-2 replacing N.

Finally, without loss of generality, we assume that 2|Z and with integer relationships

$$X^{Np-3p} + Y^{Np-3p} + Z^{Np-3p} = -Z^pG_1(X,Y) + 2Y^{Np-3} + Z^{Np-3p} = -X^pG_2(Z,Y) + 2Y^{Np-3p} + X^{Np-3p} =$$

$$= -Y^pG_3(Z,X) + 2X^{Np-3p} + Y^{Np-3p},$$

we see

$$\left(\frac{X^{Np-3p} + Y^{Np-3p} + Z^{Np-3p}}{2}, \quad XYZ\right) = 1.$$

Thus, if N be any odd number satisfying $(N(N-2), pXYZ) = 1$, from the preceding observations, we have

$$\frac{(X^{Np} + Y^{Np} + Z^{Np})}{(XYZ)^p}, \quad \frac{-((XY)^p + (YZ)^p + (ZX)^p)(X^{Np-2p} + Y^{Np-2p} + Z^{Np-2p})}{(XYZ)^p}, \quad \frac{X^{Np-3p} + Y^{Np-3p} + Z^{Np-3p}}{2}$$

are all coprime to XYZ.

We have thus lost X,Y and Z in these relationships– somewhat akin to dividing by zero.
Denoting these terms by $I_1(p,N), I_2(p,N)$and $I_3(p,N)$, we have an unbounded number of distinct relationships $I_1(p,N) - I_2(p,N) = 2I_3(p,N)$ where $I_1(p,N), I_2(p,N)$and $I_3(p,N)$ are odd and coprime to XYZ.
i. e. $X^p + Y^p + Z^p = 0$ in arithmetic may be used to generate unbounded distinct relationships coprime to XYZ.
In the case p=1 the recurrence relationship (1) is unconditional based on the collection of solutions X(u,v)=u, $Y(u,v) = v$ and $Z(u,v) = -(u+v)$ and the unbounded relationships being discussed are driven inductively from the undisputed truth that $u + v + \big(-(u+v)\big) = 0$.
For odd prime p we either have a corresponding family of solutions X(u,v,...), Y(u,v...) and Z(u,v...) which when substituted into the recurrence relationship make it an unconditional identity or we have an unbounded collection of distinct relationships with the values of X,Y and Z unknown.

The conditions: $\forall$ odd N: $(N(N-2), pXYZ) = 1$ and $\forall$ prime q: $v_q\{I_1(p,N)\} = v_q\{I_2(p,N) + 2I_3(p,N)\}$ where $v_q(M)$ denotes the precise power that the prime q divides M, provide a candidate, unbounded collection of propositions lacking inductive mechanisms to help collapse the collection to an equivalent finite collection of propositions in arithmetic. i.e. without X,Y Z to work with, we need to examine possible X,Y and Z in order to satisfy the conditions and each prime q needs to be considered separately.

We are thus able at this stage to reject the idea of a stand alone counter example to FLT(p) which is unrelated to a family of solutions because the verification of the assumed solution is in principle a simple piece of addition and multiplication. It cannot be used to generate an unbounded number of logically distinct results in arithmetic (UD1).

So we either have a parameter family of solutions for FLT(p) or FLT(p) is unprovable in arithmetic and no counter examples exist to be found either in practice or theory.

Another approach using multiplicative independence of terms

In the earlier second note we looked at ways of bypassing the confounding fact that X+Y+Z = 0 has many non-trivial solutions in arithmetic, by considering tower type numbers p(K) defined by p(1)=p and p(K+1)= $p^{p(K)}$ for K=1,2…… . For odd prime p the p(K) are all different but for K=1 they are all the same. We may then consider the same relationship

$$\frac{(X^{p(K)}+Y^{p(K)}+Z^{p(K)})}{(XYZ)^p} = -((XY)^{2p}+(YZ)^{2p}+(ZX)^{2p})\ \frac{(X^{p(K)-2p}+Y^{p(K)-2p}+Z^{p(K)-2p})}{(XYZ)^p} +$$

$$+(X^{p(K)-3p}+Y^{p(K)-3p}+Z^{p(K)-3p}),$$

but in this case one or all of the terms may have a common factor with XYZ.

Expressed in the previous notation, this relationship is $I_1\left(p,\frac{p(K)}{p}\right) = I_2\left(p,\frac{p(K)}{p}\right) + 2I_3\left(p,\frac{p(K)}{p}\right)$.

For a more obvious causal explanation we may wish to look at divisor properties of the three terms which forces the outcome $((XY)^{2p}+(YZ)^{2p}+(ZX)^{2p}) = 0$, thus providing a contradiction.
For example a prime q:(q, XYZ)=1 and a monotone increasing sequences of natural numbers $\{a_n\}$ and $\{K_n\}$ satisfying:

$$q^{a_n} | (X^{p(K_n)}+Y^{p(K_n)}+Z^{p(K_n)}) \text{ and } q^{a_n} | (X^{p(K_n)-3p}+Y^{p(K_n)-3p}+Z^{p(K_n)-3p})$$
$$\text{but } q^{a_n} \nmid (X^{p(K_n)-2p}+Y^{p(K_n)-2p}+Z^{p(K_n)-2p}).$$

The idea is to locate a proposition which is either true or unprovable in arithmetic (UD1).

Notes

UD1 is the axiomatic arithmetic of the rational numbers without the necessary additional assumptions required for real and complex analysis. The well known axiomatic system UD1 is discussed in : '*Euler's constant and the Riemann hypothesis*' –www.peterbraun.com.au

We do not use the phrase 'Peano arithmetic' here because some work includes all standard complex analysis in the meaning.

For the case N=2 there is a complete family of solutions in the Gaussian integers.

A more motivated approach to the first section may be useful:

$$\text{Let } R(X) = \prod_{1\le k\le N}(X - X_k), \qquad P(n) = \sum_{1\le i\le T} X_i^n, \text{(fixed } T \le N),$$

For $1 \le j \le T$ let e_j be the j^{th} elementary symmetric polynomial of $X_1, X_2 \ldots X_T$, with $e_0 = 1$.

Set $X_i = 0$ for $T + 1 \le i \le N$.

Then

$$0 = \sum_{1\le i\le N} R(X_i) = \sum_{0\le r\le T} (-1)^r P(N-r)e_r.$$

Hence for $T = 3$, we have

$$P(N) - P(N-1)e_1 + P(N-2)e_2 - P(N-3)e_3 = 0.$$

With $X^p = X_1, Y^p = X_2$, $Z^p = X_3$ and $X^p + Y^p + Z^p = 0$ this reduces to

$$6P(Np) - 3P(2p)P(Np-2p) - 2P(3p)P(Np-3p) = 0.$$

Another application of these ideas about the limitations of arithmetic is to be found in the theory around the Riemann zeta function.
Here we have UD1 (arithmetic) and UD2 (complex analysis) as two separate axiomatic systems which are closely related. There are propositions in UD1 very closely related to the positioning of the zeros of ζ in the critical strip.

Let $M_K(x) = \sum_{n\le x} M_{K-1}(n)$, where $M_1(x) = \sum_{n\le x} \mu(n)$ and μ is the Möbius function.

The big O growth rate of the Mobius sum function and the higher sums and also the corresponding limits on oscillatory behaviour, both positive and negative, provide propositions equivalent to the Riemann hypothesis.

If the minimum oscillation bands $\Omega_{+-}\left(x^{K-(1/2)-\epsilon}\right)$ for $M_K(x)$ (K=1,2,3…) are unprovable in UD1 then no wider oscillation bands are possible to prove by calculation. RH follows.