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A special class of number theoretic functions - Peter Braun

Let $\upsilon(N, p)$ denote the highest power of p dividing N.

We write a Є A if a is a number theoretic function constant on forms satisfying

$(1/ \upsilon(N, p)) \{a(N/p) + a(N/p^2) + \ldots + a(N/p^{\upsilon(N, p)}) \} = F([N])$ ....(1)

for each prime p dividing N.

We see many of the Dirichlet series which appear in discussions in the theory of the Riemann zeta function have coefficients which satisfy these conditions. Indeed the Riemann zeta function may be considered the simplest function with coefficients satisfying the conditions.

We note that with F and a satisfying (1) we have the relationship

$\sum F(n)\log(n)/n^s = -(\zeta'(s)/ \zeta(s))\sum a(n)/n^s \quad (a(1) = 0)$.

The *Algebra of number forms* includes the ordering of forms [n] for which $\upsilon(n) = K$ (K 1,2,3…).

(Note this function is not to be confused with $\upsilon(N, p)$ as defined above)

We call the collection of forms with $\upsilon(n) = K$ the class of forms with index K.

It is an interesting property of the functions being studied here that the values on the function are determined if the values on one form in each class are specified.

To get an idea of why this is true we look at a simple numerical development:

Different letters denote different primes.

Let $n = p^2q$. Then $½\{a(pq) + a(p)\} = a(p^2)$.

With a(p) specified this last equation determines one of a(pq), $a(p^2)$ from the other.

Let $n = (p^3q)$

Then $1/3\{a(p^2q) +a(pq) +a(q) \} = a(p^3)$

Also with $n = p^2qr$ we have

$½\{a(pqr) + a(qr)\} = a(p^2q)$.

Clearly specifying one of a(pqr), $a(p^2q)$ and $a(p^3)$ determines the values of the other two.

We first prove a special case of this observation.

Let P_N denote a product of N distinct primes.

Theorem 1

For all a Є A and all forms [n] there exist non negative rational $\lambda_i^{[n]}$ $(1\leq i \leq \upsilon([n])$ such that $a(n) = \sum \lambda_i^{[n]} a(P_i)$ where $\sum \lambda_i^{[n]} = 1$. Each summation is $1\leq i \leq \upsilon([n]$.

In the above illustration, we have

$a(p) = 1a(p)$

$a(pq) = 0a(p) + 1a(pq)$

$a(p^2) = (1/2)a(p) + (1/2) a(pq)$

$a(pqr) = 0a(p)) + 0a(pq) + 1a(pqr)$

$a(p^2q) = 0a(p) + (1/2)a(pq) + (1/2)a(pqr)$

$a(p^3) = (1/3)a(p) + (1/2)a(pq) + (1/6)a(pqr)$

$a(pqrs) = 0a(p)) + 0a(pq) +0a(pqr) + 1a(pqrs)$

$a(pqr^2) = 0a(p)+0a(pq) +(1/2)a(pqr) + (1/2)a(pqrs)$

$a(p^2q^2) = 0a(p) + (1/4)a(pq) + (1/2)a(pqr) + (1/4)a(pqrs)$

$a(p^4) = (1/4)a(p) + (11/24)a(pq) + (1/4)a(pqr) + (1/24)a(pqrs)$

....

The proof of this continuing pattern will be by induction.

Let $T_1 = \{[p]\}$, $T_2 = \{[pq], [p^2] \}$, $T_3 = \{[pqr] , [p^2q],[p^3]\}$ with $T_K = \{[n] : \upsilon(n) = K\}$.

The inductive proof will take the ordering of the classes in each T_K into account.

We firstly establish the existence of such a representation and then prove the uniqueness.

We see the representation is possible for T_1 T_2 and T_3.

Now assume such a representation is possible for each of the classes in

T_1, T_2, … T_N. The smallest element of $T_{(N+1)}$ is the class defined by a product of N+1 distinct primes and we note

$a(P_{(N+1)}) = 0a(P_1) + 0a(P_2) + ...0a(P_N) + 1\, a(P_{(N+1)})$.

Hence the representation is possible for this element of $T_{(N+1)}$.

Let [n] be any other element of $T_{(N+1)}$ and assume the theorem is true for [m] <[n]. There must be a prime p such that p^2 |n and let $\upsilon(p,n) = t$. Let P be a prime which does not divide n. Then

$a(n) = a(Pn/P) = (1/t)\{ a(Pn/p)+ a(Pn/p^2)+ ....+ a(Pn/p^t)\}$ ……………..(1).

We note each of $[Pn/p^2]$ … $[Pn/p^t]$ is less than [n].

Now $\upsilon([Pn/p]) = \upsilon([n])$ but Pn/p has more primes q to the power 1 than n and hence from the way the ordering is defined [Pn/p] < [n] .

Thus by the inductive assumption each of a(Pn/p), $a(Pn/p^2)$,.... $a(Pn/p^t)$ has a representation of the type given in the statement of the theorem.

If these representations are substituted into (1) with like terms gathered together we necessarily obtain a corresponding representation for a([n]).

We now show that this representation is unique.

We use the following:

Lemma1

For $k \geq 1$ let $d_k(n)$ be defined by $\sum d_k(n)/n^s = \{\zeta(s)\}^k$. Then $d_k \in A$.

Proof

For $\sigma > 1$ we have

$k\{\zeta(s)\}^{k-1} \zeta'(s) = -\sum d_k(n)\log n/n^s$.

We may write this as

$k\{\zeta(s)\}^k\{ \zeta'(s)/ \zeta(s)\} = -\sum d_k(n)\log n/n^s$.

Comparing coefficients, using the von Mangoldt function Λ,

$-k\sum d_k(N/g)\Lambda(g) = -d_k(N)$ where summation is over the divisors g of N.

Since the logarithms of primes are linearly independent over Q the result follows.

The uniqueness

Suppose now for some [n] > [p] we have two representations $a([n]) = \sum\lambda_i a(P_i)$ and $a([n]) = \sum\lambda'_i a(P_i)$ for a Є A and these are different where both sums are finite.

Then $= \sum(\lambda_i - \lambda'_i)\, a(P_i) = 0$ has non- zero coefficients .

If we choose $a = d_k$ we note $d_k(P_i) = (d_k(p))^i$ but then the polynomial equation $\sum(\lambda_i - \lambda'_i)\, x^i = 0$ has unbounded solutions $d_1(p), d_2(p), ...$

i.e. $\sum(\lambda_i - \lambda'_i)\, x^i = 0$ for x = 1, 2, 3 ....., and this is not possible.

Theorem 2

Let $a(P_1), a(P_2), \ldots a(P_N)$ ....be assigned complex number values and let a([n]) be assigned values according to the coefficients in Theorem 1. Then a Є A.

Proof

The uniqueness of coefficients in Theorem 1 essentially implies this.

No matter how the expressions for the a([n]) are derived there is only one form for the final result.

Thus, since the values of $a(P_1), a(P_2), \ldots a(P_N)$ .... define the function a([n]) for each n, every single equation derived from the equations defining a Є A will be consistent with the uniqueness and since this consistency is the total requirement for a Є A it follows that defining values on the $a(P_n)$ is sufficient to construct a function a Є A.

Now with $H_1 = \{[p]\}$, $H_2 = \{[pq], [p^2]\}$, $H_3 = \{[pqr], [p^2q], [p^3]\}$, ...

we prove the generalisation of theorem.

Theorem 3

Let $Q_1, Q_2, Q_3 \ldots Q_N$ be selected by choosing exactly one element from each of $H_1, H_2, H_3, \ldots H_N$ and a Є A then the values of a([n]) with $\upsilon(n) \leq N$ may be expressed as unique linear combinations of $a(Q_1), a(Q_2), \ldots .. a(Q_N)$ with rational coefficients where the linear combinations are independent of a Є A. Further the sum of the coefficients equals unity.

Proof

We easily see the theorem is true for all combinations for n= 1, 2 and 3.

We assume the same is true for n = 1, 2 …N and examine the case n = N+1.

We order the elements of $H_{(N+1)}$ as $Q_1, Q_2 \ldots .Q_N, Q_{(N+1)}$ using the multiplicative ordering of forms. Recall that Q_1 is defined by the product of N+1 distinct primes and each other Q_r is divisible by a p^2 in the form definition.

Suppose for now that the truth of the theorem has been established on Q_r for r = 1,2, K-1 where K > 2. (The case K = 2 will be proven separately).

Then let $m = PQ_K$ where we assume P is co-prime to Q_K and $p^2|Q_K$.

Using the fundamental property of a on m we thus have

$a(Q_K) = (1/\upsilon(m,p))\{a(PQ_k/p) + a(PQ_k/p^2) + \ldots + a(PQ_k/p^{\upsilon(m,p)})\}$.

The arguments of the a function on the right hand side define forms which are all smaller than Q_K. The result then follows for Q_K.

To complete the inductive proof we need to establish the starting point:- that the theorem holds for Q_1.

We only need consider the case where Q_1 is the product of N+1 distinct primes.

Since representation is unique in theorem 1 we may write

$a(Q_1) = \lambda_1 a(P_1) + \lambda_2 a(P_2) + \ldots + \lambda_{(N+1)} a(P_{(N+1)})$.

Then $a(P_{(N+1)}) = (1/\lambda_{(N+1)})\{ a(Q_1) - \lambda_1 a(P_1) + \lambda_2 a(P_2) + \ldots + \lambda_N a(P_N)\}$

and each of $a(P_1)$, $a(P_2)$, ...$a(P_N)$ is expressible using elements from $H_1,H_2,...H_N$.

In the manipulation of linear expressions and substituting linear expressions we are always using coefficients which sum to unity. It follows that the sum of coefficients in the unique representation is unity.

Although we have many choices for a 'basis' to describe an a function and these can be ordered in an obvious way, there are only two which admit to easy description.

Namely $a(p_1)$, $a(p_1p_2)$, $a(p_1p_2p_3)$ ...... and $a(p^1)$, $a(p^2)$, $a(p^3)$....

The series $1/\zeta(s)$ has an a function defined by $a(p_1p_2...p_n) = (-1)^n$ and the series for ln $\zeta(s)$ has an a function defined by $a(p^n) = 1/n$.

Notes

H_1, H_2 and H_3 have 1, 2, and 3 elements respectively. However for $N>3$, H_N has more than N elements. Thus with $T_1, T_2, ...T_M$ denoting the elements of H_N and $a(T_r) = c_{r,1}a(P_1) +c_{r,2}(a(P_2) + ...+ c_{r,,N}\, a(P_N)$ ($1 \leq r \leq M$ with $M>N$) we have inter-relationships between the $a(T_r)$.

For example, a little computation reveals

$4a(p^2q^2) -3a(pq^3)-a(pqr^2) \equiv 0$.

For the divisor function we have 4.3.3 – 3.2.4 – 2.2.3 = 0 which on cancellation gives

2.3 – 4 – 2 = 0. It would be interesting to understand the nature of the 'skeleton' numerical relationships.

We now extend the result of lemma 1 to integer k.

It turns out that if a, b Є A and c is defined by normal Dirichlet series multiplication $c(n) = \sum a(g)b(n/g)$ with summation over all divisors of n, then c Є A. It does not seem to be easy to derive this directly from the definition of a and b but such a proof would eliminate the need for the following lemma.

Lemma 2

For each natural number i let c_i Є A and have the property that for each form [n] there exists $K_{[n]}$ such that $c_i([n]) = 0$ for $i > K_{[n]}$. Let $c(n) = \sum c_i(n)$ where summation is $1\leq i \leq K_{[n]}$. Then c Є A.

Proof

We note the property a, b Є A → a+b Є A where $(a+b)(n) = a(n)+b(n)$.

The lemma then follows immediately from (1) since for any nominated [n], c([n]) is a finite sum of $c_i([n])$.

Theorem 4

For each $k \geq 1$ let c_k be defined by $(\zeta(s)-1)^k = \sum c_k(n)/n^s$. Then c_k Є A.

Proof

This follows directly from lemma 1 and lemma 2 using the binomial expansion of $(\zeta(s)-1)^k$.

Theorem 5

Define $d_k(n)$ for each integer k by $\zeta(s)^k = \sum d_k(n)/n^s$. Then d_k Є A.

Proof

We note the d_k are well defined and this result extends lemma 1.

For positive k, $1/\zeta(s)^k = 1+((\zeta(s)-1))^{-k}$

$$= \sum {}^{-k}C_r(\zeta(s)-1)^r.$$

The result follows using lemma 2.

We note with $a \in A$ that the formal Dirichlet $\sum a(n)/n^s$ may not have a half plane of convergence. For example if we let $P_n = p_1p_2....p_n$ be the product of the first n prime numbers and let $a([P_n]) = \exp(P_n)$, we note $a(P_n)/P_n^{\sigma}$ prevents convergence for any σ.

In the next section we thus consider formal Dirichlet series without concern for a half plane of convergence although the criteria for this condition will also be examined.

Similarly we will consider formal power series where initially we will not be concerned with convergence.

Theorem 6

Let λ_n be defined complex numbers for $n \geq 0$ and let f be the formal power series defined by $f(z) = \sum \lambda_n z^n$.

Let $\sum a(n)/n^s$ be the formal rearrangement of $f(\zeta(s)-1)$ as a Dirichlet series.

Then $a \in A$.

Proof

Expanding each term in the power series and using lemma 1 and lemma 2 the result follows.

The converse of this theorem is true and provides an explanation of the underlying structure of the $a \in A$.

Theorem 7

Let $a \in A$. Then there exist $\lambda_0, \lambda_1, \lambda_2$ ... such that with $f(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + ...$,

$f(\zeta(s)-1)$ rearranged as a Dirichlet series equals $\sum a(n)/n^s$.

Proof

We match up coefficients consistently in the identity $f(\zeta(s)-1) = \sum a(n)/n^s$.

Firstly note that the number $1/(p_1p_2...p_n)^s$ only appears in the terms of $(\zeta(s)-1)^n$ and matching coefficients we require $a(p_1p_2...p_n) = \lambda_n(n!)$.

We use this to define $a(p_1)$, $a(p_1p_2)$, ...

Then define a Dirichlet series b(n) by

$\sum b(n)/n^s = f(\zeta(s)-1)$ where $f(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + ...$

We note $b \in A$.

Then by the uniqueness in Theorem 1 we necessarily have $a(n) = b(n)$.

We are now in a position to give a simple proof that if $a,b \in A$ and $c = a*b$ is defined by Dirichlet series multiplication then $c \in A$.

Indeed if $\sum a(n)/n^s = F(\zeta(s)-1)$ and $\sum b(n)/n^s = G(\zeta(s)-1)$ then

$\sum c(n)/n^s = \{\sum a(n)/n^s\}\{\sum b(n)/n^s\} = F(\zeta(s)-1)\, G(\zeta(s)-1) = H(\zeta(s)-1)$.

The result follows using theorem 5.

The relationship between the Dirichlet series and the power series carries more information.

Theorem 8

Let $f(s) = \sum a(n)/n^s = F(\zeta(s)-1)$ where $F(z) = \sum b_n z^n$.

Then f has a half plane of convergence if and only if F is analytic at $z = 0$.

The proof of this theorem is not complicated but it is currently rather long so it will be the subject of a separate discussion.

The partial sums of the coefficients of a Dirichlet series are important because of the integral representation $f(s) = \int A(x)/x^{(s+1)}dx$ where $A(x) = \sum a(n)$ $(1 \leq n \leq x)$ and $f(s) = \sum a(n)/n^s$.

For convenience we consider a $\in$ A such that a(1) =1

Theorem 9

Generally with $f(s) = \sum a(n)/n^s$ let $S_f(x) = \sum a(n)$ $(1 \le n \le x)$.

Then for all such f and $x \ge 1$, and $n \ge 1$

$S_x(f) = 1 + \sum S_x(((\zeta(s)-1)/\zeta(s))^n)\, a(p^n)$ ....(i)

and

$Sx(f) = 1 + \sum S_x((\ln\zeta(s))^n/n!)a(P_n)$ ....(ii)

This provides an interesting connection between defining a by its value on prime powers and defining a by the values on prime products.

From the preceding discussion either may be used to define a.

The problem of translating between any 2 ways of defining a would in general involve functions of $\zeta(s) - 1$.

We note that the summations in (i) and (ii) are always finite.

Further the S_x behaves like a linear function on Dirichlet series. Consequently, it suffices to show that (i) and (ii) are true for $\zeta(s)^k$ for k = 1,2,3.....

For (i) we require $S_x(\zeta(s)^k) = 1 + \sum S_x(((\zeta(s)-1)/\zeta(s))^n)d_k(p^n)$, k = 1,2 ....

which would follow from an identity

$\zeta(s)^k = 1 + \sum d_k(p^n)\{((\zeta(s)-1)/\zeta(s))\}^n$.

To this end we prove the identity

$1/(1-X)^k = 1 + \sum d_k(p^n)X^n$ ....(3)

For k=1 this is $1/(1-X) = 1 + \sum X^n$.

Assume the identity for n = K.

Then $1/(1-X)^{K+1} = \{1/(1-X)\}\{1 + \sum d_k(p^n)X^n\}$.

$d_{(K+1)}(p^n) = d_K(p) + d_K(p^2) + ... + d_K(p^n)$ clearly follows from $\zeta(s)^{(K+1)} = \zeta(s)\zeta(s)^K$ and the identity follows.

The proof of (i) then follows substituting $(\zeta(s)-1)/\zeta(s)$ for X in (3).

To establish (ii) it suffices to show that

$\zeta(s)^K = 1 + \sum(1/n!)(\ln(\zeta(s))^n d_K(P_n)$, but we note $d_K(P_n) = \{d_K(p)\}^n = K^n$
since $d_K(p) = 1 + d_{(K-1)}(p)$ and $d_1(p) = 1$ and so this identity is just $\zeta(s)^K = \exp(K\ln(\zeta(s))$.

Notes:

The function F defined by $F(z) = \exp(z)/(1-z)$ has a local inverse H which is analytic in a neighbourhood of w = 1.

We have $F\{H(w)\} = w$ for $|w-1| < A$.

Writing $w = \zeta(s)$ and $H(w) = f(s)$ we have

$H(w) = a_0 + a_1(w-1) + a_2(w-1)^2 + ...+...$ and so

$f(s) = H(\zeta(s)) = a_0 + a_1(\zeta(s) -1) + a_2(\zeta(s) -1)^2 + ...+...$ for $|\zeta(s) -1| < A$.

We thus see using Theorem 8 that f(s) is a Dirichlet series with a half plane of convergence and if a is the function defining the coefficients then a $\in$ A.

But $F(f(s)) = \exp(f(s))/(1-f(s)) = F(H(\zeta(s)) = \zeta(s)$ for $|\zeta(s) -1| < A$.

i.e. $(1-f(s))\,\zeta(s) = \exp\{f(s)\}$.

It is not obvious from this relationship that f(s) has a half plane of convergence and the current proof of this depends on Theorem 8.

Calculation of coefficients a(p), a(pq), ... involves an interesting recurrence relationship.

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In fact with $x_n = a(P_n)$ where as usual P_n is the product of n distinct primes, then for $n \geq 2$ we have

$2x_n + x_{(n-1)} = (1/2)\sum {}^N C_r x_r x_{(N-r)}$ (summation $1 \leq r \leq N-1$)

If we write $x_N = (N!)q_N / 2^{(N-2)}$ this relationship becomes

$q_N + (q_{(N-1)}/N) = \sum q_r q_{(N-r)}$ (summation $1 \leq r \leq N-1$).

Interest in this function preceded the above discussion.