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Naïve Sieve Theory – by Peter Braun

Introduction:

Naïve sieve theory is not strictly speaking sieve theory because the immediate goal is not about deducing results. It is about supporting the notion of explanation and looking at simple conditions on the sieve of Eratosthenes which provide possible 'causal' type explanations of theorems. In this sense it is different from results type number theory and also different from probabilistic number theory. But it is also about searching for unprovable statements which may be used as an underlying cause for things we would like to prove. In a way this may be viewed as a way out of some of the problem solving methods which have tried to deduce answers using manipulation backed by extraordinary ideas. These methods do seem to have limits. Chen's theorem is tantalisingly close to the end result for Goldbach's conjecture but the parity problem could be an obstacle which cannot be moved around. In this area of problem it seems more important that 2 is roughly 2 and 3 is roughly 3 and 5 is rouhly 5 etc. This is not at all the case with the Riemann hypothesis where we only need a certain density of primes and it is not at all important how many of the early ones are left out. Thus we look for global statements which provide results and then wonder if there is some way we will be able to explain the statements as unprovable.

The mechanism is simply A (unprovable), $A \rightarrow I$ (true).

It is not suggested that the following A's are unprovable (or even true) but we are dealing with I's which look as if they should be true and it is believed that this approach cannot be ruled out as a method of proof. We should look for A's which are unprovable and I's which are interesting. Ideally, we would have logical equivalence between A and I.

The discussion below provides naïve explanations for some of the well-known questions of interest.

Naïve Theorems

Usage and notation:

Let B be any set of integers. By a+pB we mean the set $\{a+pb: b\epsilon B\}$.

A set of numbers A will be said to be covered by a set of distinct primes P if each element of A is divisible by at least one prime in P.

If p_i is the smallest prime dividing n we say p_i is a significant multiple of n.

Let P_n denote the set of the first n prime numbers.

(a) Let T_n be the maximum number such that T_n consecutive numbers are covered by P_n

(b) Let S_n be the maximum number such that the sets a_1+p_1Z, a_2+p_2Z… a_n+p_nZ combined, include the numbers 1,2, 3… S_n, where a_i is a residue modulo p_i for $1\leq i \leq n$, and Z denotes the integers.

Lemma

$S_n = T_n$.

Proof

To this end consider a maximal set of consecutive numbers M+1, M+2, … $M+T_n$ which are covered by P_n.

For each prime number p_i select the smallest $r_i \geq 1$ such that $p_i \mid M+ r_i$.

We may assume r_i exists amongst the numbers 1, 2, T_n since otherwise p_i has not been 'used' as a covering prime in arriving at the maximal sequence and we easily construct a longer sequence $K+1, K+2\ldots\ K+T_n, K+T_n+1$ in which the first T_n+1 numbers are covered by $P_n \sim \{p_i\}$ and $p_i \mid M+T_n+1$ which contradicts the maximal choice of T_n. Indeed for $j \neq i$ select $J \equiv M \bmod p_j$ and $J \equiv M+T_n+1 \bmod p_i$ using the Chinese remainder theorem. But then $J+1, J+2 \ldots J+T_n+1$ are covered by P_n.

Then the sets $r_1+p_1Z, r_2+p_2Z \ldots r_n+p_nZ$ include the numbers $1,2,3, \ ,T_n$.

It follows that $S_n \geq T_n$.

To get the inequality in the other direction let $a_1+p_1Z, a_2+p_2Z\ldots\ , a_n+p_nZ$ be a choice which produces a maximal sequence $1, 2\ldots\ S_n$ of elements in these sets.

By the Chinese remainder theorem we may choose $J \equiv -a_i \bmod p_i \ \ 1\leq i \leq n$.

But then $J+1, J+2\ldots\ J+S_n$ are covered by P_n which means $T_n \geq S_n$.

It is useful to hold the equivalence of (a) and (b) firmly in mind in considering certain problems in arithmetic.

We know that $T_n > p_{(n+1)}-2$ for $n > 5$ so that sieving using $M \equiv 0 \bmod p_i \ (1\leq i \leq n)$ does not immediately come across a maximum number of consecutive numbers covered by P_n. We need to look at larger numbers to find such a sequence.

Instead of starting p_1 on p_1, p_2 on $p_2\ldots p_n$ on p_n in the sieve of Eratosthenes we may modify the sieve and start p_i on a_i for some specify choice of residues and immediately mark out T_n consecutive numbers using the marking method of the sieve. Indeed, if $M+1, M+2 \ldots M+T_n$ is a maximal sequence and $M \equiv -a_i \bmod p_i$

$(1\leq i \leq n)$ we just mark a_i and $a_i + \theta p_i$ and the maximum number of consecutive numbers will be marked. This provides an alternative method of determining the value of T_n.

We note that a maximum run of consecutive integers covered by P_n includes significant multiples of each prime $p_1,p_2\ldots p_n$.

Goldbach's conjecture (GB)

We describe here a small extension of the above thinking with an assumption which implies GB. The idea is not to wonder whether the assumption is true but just that it provides a simple explanation in terms of a cause.

which starts at any modulus

Consider two standard number lines one below the other.

On the top line mark out numbers according to the sieve

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for the prime p_i, for $p_1,p_2 \ldots p_n$.

On the bottom line choose an arbitrary selection of residues a_i modulo p_i $(1\leq i \leq n)$ to start the marking, the only condition being that $\mathbf{p_1}$ starts in the same place on the bottom line as it does on the top line (i.e. consistent with the sieve of Eratosthenes).

What we are looking for is the maximum length away from unity before we get a miss on both lines, one above the other. Let GB_n denote this number. It is easy to see that GB_n is well defined.

Proposition 1

If $GB_n < (p_n)^2$ then every even number may be written as the sum of two primes.

Proof

It should be obvious that this is a much stronger condition than is necessary to find an equivalent statement to GB. Indeed the standard sieve of Eratosthenes is one of the 'runs' which needs to be considered to arrive at the maximum GB_n.

Hence if we apply the sieve of Eratosthenes to the two number lines one below the other, using the primes with $p_1, p_2\ldots p_T$ denoting the primes less than or equal to $\sqrt{E}$, on

1 2 3 4…….. E-1 E

2E-1 2E-2 2E-3 2E-4….. E+1 E

moving left to right on the top line and right to left on the bottom line avoiding the double count on E, we find 2 prime numbers adding up to 2E.

(For those who dislike 1 being a prime a minor variation is required).

As mentioned the idea here is not to be overly interested in the truth of the assumption but purely to appreciate a causal explanation in a simple setting.

If we wish to start from a statement logically equivalent to Goldbach's theorem we also need to take into account the primes less than or equal to $\sqrt{E}$ as these will be marked in the sieving process (Eratosthenes).

A similar assumption also implies unbounded twin primes. This may be framed in a number of ways but the original sieve provides a simple example.

We easily see that m and m+2 co-prime to P_n occurs unboundedly. Let TP_n be the minimum number such that the set $\{a_i+p_iN: 1\leq i \leq n\}$ necessarily contains two numbers m and m+2 which are both co prime to P_n, for every choice of residues $a_1,a_2...a_n$.

Proposition 2

If $TP_n < (p_n)^2$ then there are unbounded twin primes.

Proof

Since TP_n covers all possible starts for marking $p_1,p_2...$ it includes the sieve of Eratosthenes.

Thus the numbers 1, 2, 3......TP_n contain numbers m and m+2 which are not covered by the sieving process if we sieve using the primes $p_1, p_2, ...p_T$ where $p_T \leq \sqrt{TP_n} < p_{(T+1)}$. Since these numbers are co-prime to $p_1, p_2 ...p_T$ they are necessarily prime.

Dirichlet's Theorem on primes in arithmetic progressions

Dirichlet's celebrated theorem has also been proven more recently by elementary methods at the same level of strength as the prime number theorem (Gelfond and Linnik [3]).

We may also find a possible explanation for 'unbounded primes' in terms of T_n.

Proposition 3

Let (a,b) = 1 and suppose $T_n +1 < ((p_{n+1})^2 - b)/a$ for sufficiently large n.

Then the progression an+ b supports unbounded primes.

Proof

For sufficiently large n we assert the sequence

$a+b, a2+b, ...a(T_n+1) + b$ contains a member co prime to P_n.

Indeed, if this is not the case then with $aa_i \equiv -b \bmod pi$ $(1\leq i \leq n)$ we see the set

$\{a_i+p_iZ : 1\leq i \leq n\}$ contains T_n+1 consecutive numbers. But $S_n = T_n$.

So at least one of $a+b, a2+b .... a(T_n+1) + b$ is co-prime to P_n .

If it were composite we would have $a(T_n+1) + b \geq (p_{(n+1)})^2$ and this is against the assumption of the proposition. This result is also implicit in the next propositions.

The growth rate of T_n

We note there are $(p_1-1)(p_2-1)...(p_n-1)$ numbers less than P_n which are co prime to P_n and these numbers are the ones which finish runs of consecutive numbers covered by P_n. The average distance between these numbers is $K_n = P_n/(p_1-1)(p_2-1)...(p_n-1)$.

Now $\ln(P_n/(p_1-1)(p_2-1)...(p_n-1)) = \sum\ln(1-(1/p_i)) = \ln(\ln(x) +A-1 + O(1/\ln x)$ as $x \to \infty$ using Merten's theorem (Chanderesakharan [2].

Consequently, $K_n = e^A \ln x + O(1)$ as $x \to \infty$. There are many solutions to m, m+2 co-prime to P_n so the downward variation on this estimate is known. There is however no useful upper bound widely known if known at all.

The growth rate of T_n as $n \to \infty$

If we start sieving according to Eratosthenes we find $p_{n+1} - 2$ consecutive numbers covered by P_n. How much better can we do by starting our 'sieving' at different start points?

$T_n = \partial(n)\, p_n$ where $\partial(n) = O(n^{\varepsilon})$ as $n \to \infty$ is probably the very best we could ever hope for but consequences of this assumption indicate how difficult this might be to prove by elementary methods.

In which case $GB_n = \partial(n)\, (p_n)^2$ where $\partial(n) = O(n^{\varepsilon})$ as $n \to \infty$ may be enough to prove the Goldbach conjecture but not using the simple discussion earlier.

In Braun [1], the approach was taken to see what sort of results followed from assumptions about T_n and these results represent refined arithmetic which resulted from joint work with A. Zulauf..

The theorems are stated here without proof.

The results cover 3 topics

1. The least prime in an arithmetic progression
2. The unbounded number of primes in a progression
3. The distance between consecutive prime numbers.

Let J(x) be the maximum number of consecutive numbers each of which is covered by primes less than or equal to x.

Proposition 4

Let $J(x) \leq Ax^{2-\alpha}$

The least prime Dm+l satisfies $Dm+l \leq (A+1)^{2/\alpha} D^{2/\alpha}$ $((D,l) = 1, 0 < \alpha \leq 1)$.

Proposition 5

Let $J(x) \leq Ax^{2\beta/\beta+1}$.

There is a prime $Dm + l$ with $N \leq m \leq [(A+1)^{\beta} D^{\beta} N]$ $(N \geq 1,\ \beta \geq 1\ (D,l) = 1)$.

Proposition 6

Let $J(x) \leq Ax^{1+2\gamma}$ $(\frac{1}{2} > \gamma \geq 0, A > 0)$.

For $n > N_0$ there is a prime in $[[N-AN^{(1/2)+\gamma}], N]$

$p_{n+1} - p_n = O((p_n)^{(\frac{1}{2}+\gamma)})$ as $n \to \infty$.

Notes (added July 2019):-

Intuitively, since all the primes cover all the natural numbers it would be very odd if asymptotically we did not have γ=0 or γ=0+ε. i.e. if this is not the case then the primes cover all the natural numbers plus a positive proportion more. In this is the case we do get some rather familiar targets. It would also be remarkable if γ=0+ε were decidable in rational arithmetic although an upper bound for γ would seem quite possible.

References

[1] P. B. Braun, Topics in Number Theory. D.Phil Thesis. University of Waikato,1979.

[2] K. Chandrasekharan, Introduction to Analytic Number Theory. Springer-Verlag 1968.

[3] A.O. Gelfond and Yu.V. Linnik, Elementary Methods in Analytic Number Theory (Translated by Amiel Feinstein) Rand McNally & Co, 1965.