A note on the Riemann Hypothesis (ii) – Peter Braun

This note is an extension of the first note (i) and expands on the intuitive notions which were discussed in that note.

A core requirement of proof is a connectivity of ideas presented sequentially so that any conclusions may be understood to follow from the reasoning. This requirement is evident in the' line by line' proof which is almost a necessary part of logical argument. However, in situations where ideas rather than algebraic manipulation are involved it may be more important to locate the key ingredients in the explanation.

The first note suggested a link between counting and provability and indicated that the Riemann hypothesis was logically equivalent to a proposition which could never be verified. In moving from intuition in a search for a connected argument, it becomes apparent that it is important to distinguish between purely logical activity in the sense of the classical laws of logic and the mathematical constructions which are being discussed within the language of mathematics. The principal idea in this note is that if a theorem has a finite proof then the propositions which make up the essence of the proof will only ever generate a finite number of essentially different propositions. There is of course a lot in this last sentence which needs explaining and some attempt is made to clarify this assertion towards the end of the discussion. One parallel theme in this discussion is to suggest that the Riemann hypothesis is irrational in the sense somewhat like the sense that $\sqrt{2}$ is irrational. We are able to get a finite definition of $\sqrt{2}$ but we require an infinite series to describe it numerically. The equivalent notion here would be that the Riemann hypothesis does not have a finite proof. To support this theme we start by discussing the hypothesis with a view to opening up the possibility that it may indeed be an intractable problem. That is, a problem which is 'irrational' or undecidable.

The sieve of Eratosthenes provides a practical way of working out prime numbers one at a time starting from the smallest prime. It also allows for the unique factorisation of a number to be worked out as the labelled marks on a number will indicate precisely the primes involved in the factorisation.

If we then let p_n denote the n^{th} prime we see from the point of view of the developing language that the word "p_n" has quite a complicated meaning.

It denotes the smallest number whose multiplicative structure cannot be described easily in terms of the preceding primes.

Because of the more complicated patterns in the prime numbers compared to the natural numbers the order of things is to try and explain and discover properties of primes from properties of natural numbers.

We could set out to try and construct the natural numbers multiplicatively rather than the normal additive set theoretical/logical construction, working out as we go along what assumptions we need to make in order to arrive at a sequence:

$1, p_1, p_2, (p_1)^2, p_3, p_1.p_2, p_4,(p_1)^3, (p_2)^2\ p_1.p_3,$ and so on …

It becomes apparent relatively quickly that there is no easy way of constructing the next form (the successor) and that the multiplicative structure in numbers is a higher order observation best discovered after the additive numbers have been defined

If in the practical application of the sieve we note the specific primes which 'mark' a composite and we have all the information about the order of the multiplicative structures up to the composite we are able to deduce the structure of the composite.

The point here is that a lot of the information about the prime structure of some of the first N-1 numbers is required to 'deduce' the prime structure of N and this is not obvious if the structure is determined by a more efficient mechanical process.

The multiplicative structure encodes more and more information as we move from successor to successor whereas the additive structure maintains a pristine simplicity as we move from successor to successor.

The Riemann zeta function $\zeta(s) = \sum 1/n^s$ provides a link between the additive and multiplicative structure of the natural numbers via the equivalent notion $\Pi(1-(1/(p_n^{s})))^{-1}$ which then allows for discussion about distributional properties of the prime numbers in terms of the zeros of the zeta function.

But we should look closely at the two series $\sum 1/n^s$ and $\sum 1/(p_n)^s$.

It is difficult to imagine a logician or mathematician trying to derive the natural numbers from the series definition of the zeta function. We may in a sense suspect that something about the natural numbers is lost in $\zeta(s)$.

"$\zeta(s)$" is a word for an object which helps explain many things and which itself is open to explanation.

We may ask:- how much about the prime numbers is lost in the series $\sum 1/(p_n)^s$?

As the theory of the Riemann zeta function unfolded the types of questions about primes emerged which were candidates for explanations via the function.

The critical strip appeared as an object of thought in relation to $\zeta(s)$ and for over one hundred and sixty years has become a barrier to finding out more definite information about $\zeta(s)$ and about the prime numbers as a consequence. It may be that the basic relationship between the two functions $\sum 1/n^s$ and $\sum 1/(p_n)^s$ is so complicated, it limits the extent to which one function is able to explain the other.

The function $\theta(s)$ defined by $\theta(s) = \sum 1/(p_n)^s$ appears to be atomic in nature when it comes to releasing information about the zeros of $\zeta(s)$ in the critical strip. The only definite statements have come from computation of zeros of $\zeta(s)$ and although there is some theoretical work in establishing their placement it is largely a numerical investigation in the same sense that the sieve of Eratosthenes is a practical way of finding new primes.

With $\theta(s) = \sum 1/(p_n)^s$ we have the relationship

$$\zeta(s) = \exp(\theta(s) + (1/2)\,\theta(2s) + (1/3)\,\theta(3s) + \ldots + \ldots) \qquad (1)$$

We know we have the dual relationship

$$\theta(s) = \mu(1)\ln\zeta(s) + (\mu(2)/2)\ln\zeta(2s) + (\mu(3)/3)\ln\zeta(3s) + \ldots + \ldots \qquad (2)$$

and from this that $\theta(s)$ has the line a $\sigma = 0$ as a natural boundary (Titchmarsh [2]).

Besides these two algebraic relationships there is another related way of looking at the zeta function which exhibits the complexity of the relationship between $\zeta(s)$ and $\theta(s)$:

The uniqueness of factorisation of natural numbers allows for an equivalence relation where the classes are numbers with the same essential prime structure, which we call the form of the number. Furthermore, there are two obvious ways of ordering forms.

The first is to order according to the first occurrence of a number in the class which may be called the multiplicative ordering.

The second ordering, which we list below, is the additive ordering:-

$[p]$, $[pq]$, $[p^2]$, $[pqr]$, $[pq^2]$, $[p^3]$, $[pqrs]$, $[pqr^2]$, $[pq^3]$, $[p^2q^2]$, $[p^4]$ ….

Here, moving from left to right the distinct prime factors are listed with increasing degree subject to the condition that the sum of the indices of the primes defining the form is a monotone non-decreasing sequence.

For convenience we include [1] and let

D_1, D_2, D_3, .. denote [1], $[p]$, $[p^2]$, … and let $f_i(s) = \sum 1/n^s$ where $n \in D_i$.

It should be clear that $\zeta(s) = \sum f_i(s)$ for $\sigma > 1$.

Now with $p_n(s) = \theta(ns)$ for $n \geq 1$ we regard the $p_n(s)$ as a 'prime' Dirichlet series and consider the Dirichlet series formed by products of these primes with each prime appearing to some numerical degree. In other words we consider the semi-group G generated by $\{p_1(s), p_2(s), p_3(s), .. \}$. A fundamental theorem which may be proved is that each $f_i(s)$ may be expressed as a finite sum of elements of G with rational coefficients.

The relationship
$$\sum 1/(pq)^s = \tfrac{1}{2}(\{\sum 1/(p)^s\}^2 - \{\sum 1/p^{2s}\}) = \tfrac{1}{2}\{\theta(s)^2 - \theta(2s)\}$$

is the first non-trivial case This result just about follows from the right hand side of (1) assuming a sort of uniqueness of factorisation but it may be established using properties of Dirichlet series (unpublished notes).

It follows that each $f_i(s)$ has an analytical continuation into $\sigma > 0$ and in $\sigma > \tfrac{1}{2}$ the only singularities of these functions are singularities of $\theta(s)$. Thus the series for the zeta function has all these sub-series components whose analytic character in $\sigma > \tfrac{1}{2}$ reflects the character of the logarithm of the zeta function.

Even though these relationships may appear to be relatively simple algebraically the relationship between $\theta(s)$ and $\zeta(s)$ has proved to be very complicated from an analytical point of view.

If we accept that the primes carry in them complex coding about the multiplicative structure of the preceding numbers then there may be limitations to how much information about prime numbers the theory of the Riemann zeta function will ever reveal.

Cramer's conjecture provides an example which may indicate a limitation in the capacity of the theory around the zeta function to explain a problem. In looking at the distance $(p_{(n+1)} - p_n)$ we are thinking about an additive question (on the subscripts) within an essentially multiplicative realm. With $p_{(n+1)} - p_n = O(p_n^{(C+\epsilon)})$ it may be that there is a limit to how much will ever be proved about C. Whereas C=0 is a possibility, only $C = \tfrac{1}{2}$ follows currently from the assumption of the Lindelöf hypothesis (the truth of which follows from the Riemann hypothesis).

The complexity of the prime numbers may make it impossible to distinguish between the lines $\sigma = 0$ and $\sigma = \tfrac{1}{2}$ in the deductive reasoning of complex analysis applied to the Riemann zeta function.

It may just be that in the context of this problem these parallel lines behave like the parallel lines of projective geometry and the visualisation that they 'meet' at 'infinity' has the interpretation that it is impossible to distinguish between $\sigma = 0$ and $\sigma = \tfrac{1}{2}$.

This imagery fits in nicely with the zero free regions in the critical strip, which are hard fought for, in order to derive the prime number theorem using complex variable theory.

If we entertain the possibility that the Riemann hypothesis is undecidable by a deductive process in complex analysis or any realm which includes complex analysis as a special case, then where will we find an explanation which will be acceptable to modern number theorists? Research in mathematics typically moves outwards creating new concepts and structures which improve the explanatory power of the subject but at the same time introduce new problems because there is more to think about.

The inward journey, examining the foundations of mathematics, seems more content with getting explanations for areas of mathematical activity at a very elementary level without getting into specific mathematical problems. We will argue in the following lines that one needs to look in both directions in order to understand an explanation of the Riemann hypothesis.

We see logical inquiry explaining mathematics rather than solving specific mathematical problems, whereas the working mathematician falls back on classic logic and consensus proof. The Zermelo-Fraenkel axioms may provide a tidy way of explaining how thought has developed in explaining and justifying much mathematical activity but these sorts of approaches are not necessary (in the mathematical sense) for mathematical activity to proceed. The saying that 'the proof of the pudding is in the eating' translates quite well in the mathematical context to 'the proof of the proof is in the thinking'.

The discussion is intended for those people who recognise proof in mathematics. That is, a proof is accepted if the reader is satisfied that no counter examples to any assertions will be possible and the explanation is understandable.

Logical thought is simply a part of cognitive apparatus. A person could of course believe that they have read a valid proof only to find at a later point some incompleteness or logical error or some such thing.

A person working in a language trying to explain new things in that language moves in the directions that the language permits. If barriers arise, new words are sometimes introduced to lift the explanatory level of the language. These words and the underlying concepts and principles may be drawn from other defined languages. Once identified though the initial formal language absorbs the new concepts and principles as part of the formal language or discipline.

The sequence of words for the prime numbers "p_1", "p_2" ... is a good example of introducing an unbounded number of words whilst remaining within mathematics. Once the words "1", "2", ... have been given meaning, we can define a meaning for numbers "p_1", "p_2", and so on.

The potential for sensible and essentially different conversations is seemingly unlimited.

If we are able to derive a theorem S from a theorem T we say S implies T and that S is sufficient for T. This is no more and no less than the mechanism of modus ponens which is one of the natural laws of human thought applied within the language of mathematics.

There are two areas in this examination which will now be discussed with a view to explaining that the Riemann hypothesis may be taken to be true because it is undecidable:

- Clarification of the usage of "infinite" in mathematical usage
- The status of unbounded logical chains

The author has innocently wandered into this area only to find an old battleground of views and thoughts set off by Cantor set theory and the axiom of the infinite set as prescribed in the Zermelo-Fraenkel axiomatic suite. It has not appeared necessary to adopt any particular formal system or complicated philosophical view because the Riemann hypothesis has emerged as quite a practical problem. The zeros of the function may be calculated one at a time and so far they all lie on $\sigma = \frac{1}{2}$. If this always happens then the Riemann hypothesis is true, otherwise it is false.

The theory around the Riemann zeta function is in the heartland of classical complex analysis and the axiomatic systems which support such a problem are tried and tested and accepted by many mathematicians. A defence for the sensibility of things is via the Zermelo-Fraenkel axioms, with the support and acceptance of classical logical thought as an inevitable way of reasoning in the language.

But there are differing views on what the word "infinity" actually stands for.

The following notes appear to align with an informal understanding of the finite and constructive views of discussion around these issues. Importantly no fundamental conflict between different views in these areas seems to arise.

A feature in the use of some languages is a desire by the user to get 'to the bottom of things.' This may be a drive to find an explanation for something or other which at the time is not immediately obvious. The level of explanation required is very much related to the individual's requirement and readiness to accommodate an explanation. In general, an explanation at one level may be incorrect or incomplete when included in the collection of existing explanations but the existence of such a hierarchy is very much connected with increased understanding.

The working mathematician and the logician probably find common ground in Zermelo – Fraenkel set theory in terms of the superficial appearance of the axioms. And this is genuine common ground. However, the logician or philosopher may be thinking about things in quite a different way from the mathematician who may be trying to do mathematics.

There are of course schools of logic which have different degrees of difficulty solving mathematical problems and this activity crosses the border between mathematics and logic. This orientation is really more concerned with discovering structures within the reasoning process rather than the narrower focus of mathematical problem solving. The activities are related as some people work in both fields and mathematical components are needed in order to compare the different logic systems.

The logical formality which supports mathematical activity is not quite as formal (detached) as one might wish however. No matter how much things are developed in an elemental way, in the step by step progress to explain results, there is inevitably a thinker doing the thinking or a rule follower doing the rule following, or some such thing, and the most atomic forms of formality are inter-related with thought and the thinker is obliged to use the classical laws of logic.

If the pages of the disciplines are open to view the chances are the hieroglyphics would be on their way to explaining something or other by metaphor, logic or any other imaginative way that explanation is able to be communicated.

When the notion of truth or falsity arises in a mathematical context the value ascribed to new theorems derives integrity from the explanation.

There is really a sort of democracy operating within whatever framework has been agreed upon but a fairly basic requirement is that the result follows 'logically' or 'intuitively' in some sense or other. Sensibility is established when a number of experts agree that there is something tangible being developed albeit in some realm which may only be intuitively understood. The mathematician follows the rules he believes in but leaves the door open for fresh interpretation.

Cantor set theory is like this and the development of so much mathematical thought is testimony to the extraordinary explanatory power coming from a relatively compact base.

Classical logic is seen simply as the language implicit in any language which is describing or explaining action.

Infinity

Mathematics uses infinity in two ways and it is unfortunate that this word is also in popular usage. The mathematical constructs which illustrate unbounded or never-ending, align quite well with some popular usage of the word "infinity".

At the more questionable end of things there is a desire to locate some sort of entity which is being talked about, a sort of tangible extension of unboundedness which captures this elusive notion. Mathematics copes with this using the axiom of the infinite set.

The adjective infinity:

In mathematics the Zermelo – Frankel axiom for the infinite set uses "infinite" as an adjective and the axiom provides the mechanism for producing a set containing the natural numbers. There is nothing infinite going on here in later applications to mathematics.

The set of natural numbers may be pictured as a big cupboard and if you open the door it is possible to select any natural number and this is indicated by $n \varepsilon N$. If we adhere to the usage, N contains any nominated natural number, we avoid the dissonance associated with saying N contains all natural numbers. This is quite important if we wish to avoid discussions where conflicting imagery produces different sides of an argument in talking about infinite sets:

If A pictures a set via a jam jar full of jelly beans then the notion of an infinite set becomes difficult for A. The cupboard picture is going to be more appealing as a never ending supply of jelly beans is available without the need to keep increasing the size of the cupboard. The suggestion is that the cupboard picture of set is all that one requires in the axiom of infinity except that it has to be agreed that a primitive pattern like 1, 1+1, 1+1+1, … may be used to visualise an indefinite number of distinct objects.

The noun infinity?

There is a need for clarification here. Usage, $N \to \infty$, is invariably involved with some sort of quantities P(N) which do (or don't) behave in a certain way for nominated numbers. There is no infinity here, just usage of the word "infinity". It is one of those cases where usage doesn't create an entity.

We would be better off describing the interpretation of $A_1+A_2+A_3+ \ldots..+. .$and leave it at that. We know for example from p-adic analysis that $A_0+A_1P+A_2P^2 +\ldots+\ldots$ has an altogether different interpretation. We do not need any more than the cupboard type 'set' in the infinite sets found in complex analysis. The definitions of limit and so on proceed quite normally.

The construction of complex analysis proceeds quite happily knowing that the natural numbers are accessible – we do not need to know that they are all contained somewhere. It is superfluous imagery.

This does not imply an attack on Cantor set theory which goes on to develop the theory of infinite sets. Modern set theory has proved to be a very popular way of discussing mathematics but no notion of the metaphysical infinity is necessary. There is no conflict between the finite and the infinite approach and any query should resolve as confusion in imagery around language use.

The proof by induction infinity

Peano arithmetic and the underlying set theoretical justification have led to the usage 'And hence P(n) is true for all natural numbers'. What we really need to know is that P(n) is true for any nominated natural number and that numerical investigation will never locate an exception. The cupboard view has no need for the infinite set but there is a need to accept that recognisable pattern is capable of unbounded extension – more to do with cognition than logic.

The use of 'for all N' is really as weak as – we won't find contradiction by saying 'for all' so we may as well say it. But it is not legitimising anything.

The use of 'for all' seems mainly to be a convenient way of turning the development of logical systems into an algebraic activity.

If we want to look at $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, $P(3) \rightarrow P(4)$, $P(4) \rightarrow P(5)$, …we are looking at an unbounded sequence of arguments and modus ponens and the other classical laws are only understood in the finite. The assertions $P(n) \rightarrow P(n+1)$ may be logically associated finitely.

Consider then the two existence images:-

- There exists τ which is an inductive number but not a counting number

and

- P(n) is true for all natural numbers

Both are useful mental props and props in language use for giving accounts and explanations of what is going on. We can't really think about all natural numbers and we can't comment with any degree of confidence on unbounded arguments.

We see in the axiom of infinity an element which is regarded as necessary in the formal construction of the natural numbers. It is difficult to see why it is necessary. The cardinality of the natural numbers and the infinite set interpret the extension of unbounded in one way and τ interprets the extension of unbounded in another way. It may be useful to think of the first interpretation as the numerical path and the later interpretation as the logical path.

The important difference is that the infinite set requires an axiom to be meaningful but τ is a device to bridge our thought from knowledge of the experience of counting in the external world across to the theoretical construct of the natural numbers which are contemplated in the inner world of thought.

We may transfer a notion of theoretical counting into the thought realm using an example like the grains of sand on a beach. We may be unable to count them but we are content with the idea that there are only a finite number of grains. In the finite frame of mind, the count of propositions we may take out of our proposition cupboard must be capable of being enumerated in a theoretical count since this is all we admit. It is at this point that the use of infinity in mathematical constructs is not accepted when it comes to propositions. We may contemplate an unbounded sequence of distinct propositions in an artificial way using the natural numbers, but in reasoning, the collection of possible distinct propositions which may be selected must admit to a theoretical count. Then $\{N, \tau\}$ is just recognition of the finite nature of argument.

Consequently, if we construct an unbounded sequence of logically equivalent propositions each of which is different from the others then each is undecidable. If they were provable we would be admitting (logically) a 'cupboard' collection of distinct propositions outside of a theoretic count.

Consider then such a collection of propositions, P(N), which has been proven by mathematical induction, where P(N) and P(N+1) are logically different propositions. We note that in this context $P(N) \Rightarrow P(N+1)$ and $P(N+1) \Rightarrow P(N)$. From our unbounded sequence of logically equivalent propositions the argument that P(N) is true for any N is true but the statement that P(N) is true for all N is undecidable.

Another way of getting into a frame of mind where this discussion may be taken seriously is found in seeing how we react when confronted by obstacles. The standard mathematical approach of moving around difficult states is very much a part of proof.

The classical example of $\sqrt{2}$ being irrational is a good place to start. Faced with the evidence that we cannot find numbers p, q such that $(p/q)^2 = 2$ we could contemplate that there is some false assumption in the argument leading to the evidence but instead we choose to construct a new type of number. This is creativity following a belief that an argument leading to an obstacle is sound.

Similarly, with the unbounded prime numbers, when we discover $P = (p_1 p_2 \ldots p_n) + 1$ is co prime to $p_1, p_2, \; p_n$ we don't say this is an error in our thinking – we move around the problem and decide there must be new primes. This coincides with experience and relates back to the classical inductive argument.

These sorts of manoeuvres are very much a part of mathematical activity and the phenomenon of new constructions (real numbers, new prime numbers ...) comes about through reason and creativity dealing with obstacles. Once an obstacle has been identified through reason we try to move consistently in the language to go around it, using prior experience.

In the case of the Riemann hypothesis we are of course at liberty to contemplate the problem as true, false or undecidable. The lack of proof to date is not in itself an obstacle but if we are in the frame of mind where we choose to consider why the hypothesis may be undecidable then we are trying to locate an obstacle which eliminates provability.

An explanation of the Riemann hypothesis

The Riemann hypothesis is discussed as a proposition which is logically equivalent to an unbounded sequence of essentially different propositions. This is the obstacle we move around in order to get an explanation of the Riemann Hypothesis.

Σ will mean summation $1 <= n <= X$.

Let $\{ P_1 \equiv [M_1(X) = O(X^{(1/2)+\varepsilon})$ as $X \to \infty]\}$ be the classical statement about the Möbius sum function, and let $\{P_K \equiv [M_K(X) = O\ (X^{(K-(1/2))+\varepsilon})$ as $X \to \infty$, $K >= 2]\}$, be the corresponding statement for higher sums, where $M_K(X) = \Sigma M_{(K-1)}(n)$

We know, without too much difficulty that each statement is equivalent to the Riemann hypothesis.

Indeed, more generally, with $A(X) = \Sigma a(n)$ we see that

$$\Sigma A(n) = [X]A(X) - \Sigma n a(n) + E(X) .$$

Perron's integral formula for Dirichlet series allows an inductive proof that the statements each imply RH with suitable simple estimates E(X).

The implication in the other direction involves contour integration using the Mellin transformation and the estimate $1/\zeta(s) = O(t^{\varepsilon})$ as $t \to \infty$. (See for example Titchmarsh [2]). The method is applicable for any natural number K.

Thus $P_1, P_2, ..P_n....$ is an unbounded sequence of different propositions each of which is logically equivalent to the Riemann hypothesis.
At this point we need to quantify what we mean by 'different' propositions:

There is a very clear way of supporting the assertion that

$P_1 > P_2 > P_3 \ldots$

If we start with a general sequence $\{a_n\}$, where $a_n = O(n^{\Delta+\varepsilon})$ as $n \to \infty$, we may define partial sums $A_K(X)$ in the same way the $M_K(X)$ are defined using the Mobius sequence $\{\mu(1), \mu(2), ...\}$.

That is, $A_K(X) = \sum A_{(K-1)}(n)$ where $A_1(X) = \sum a_n$ where all summation is $1 \le n \le X$.

Now let $\{Q_1 \equiv [A_1(X) = O(X^{\Delta+\varepsilon})$ as $X \to \infty]\}$ and let $\{Q_K \equiv [A_K(X) = O(X^{\Delta+K-1+\varepsilon})$ as $X \to \infty]\}$, (K>= 2) be the corresponding statement for higher sums.

Each of Q_1, Q_2, Q_3...may be thought of as being drawn from a class of propositions C_1, C_2, C_3 ….

We have $Q_1 \to Q_2$, $Q_2 \to Q_3$ … as a class property, where Q_1, Q_2, Q_3....are related according to the above definition. The reverse implications are not necessarily true. That is we may construct a case where $Q_{n+1} \to Q_n$ is false.

For example, with $A_{n+1}(N) = (-1)^N N$ $(N \ge 1)$ we do not have $Q_{n+1} \to Q_n$.

So we have an unbounded sequence of propositions each equivalent to the Riemann hypothesis where the elements in the proposition sequence are distinguishable from each other in a non-trivial way.
But a finite proof of the Riemann hypothesis would only ever generate a finite number of essentially different logically equivalent propositions.
Hence we cannot possibly prove the Riemann hypothesis directly through mathematical manipulation because it would admit a cupboard of different, equivalent propositions beyond a theoretical count.

The simplicity of the zeros

The modified Merten's conjecture $M(X) = O(X^{1/2})$ as $X \to \infty$ implies the simplicity of the zeros, Odlyzko, A.M. and Riele H.J.J. [1]. (The original result may be due to Ingham). Hence the discovery of a multiple zero would disprove the modified Merten conjecture. But $M(X) = O(X^{(1/2+\varepsilon)})$ as $X \to \infty$ is equivalent to the Riemann hypothesis and is therefore undecidable. The modified Merten's conjecture is then also undecidable. As this is the case, no multiple zero will ever be found.

An ambiguity in thought

Is this section we note there is a certain dissonance or ambiguity in how $\Pi(1-(1/(p_n^{s})))^{-1}$ is interpreted. The normal proof of $\sum 1/n^s = \Pi(1-(1/(p_n^{s})))^{-1}$ uses the notion of limit and the convergence of the difference to zero.

The terms on the left hand side of the equation are clearly countable. The product terms on the right hand side are uniquely formed and hence within analysis there are a countable number of terms. The mind goes through the process of imagining all the combinations and comes up with a countable number of terms.

We demonstrate now that the mind can go through exactly the same process and come up with an uncountable number of terms.

We may extend the natural numbers to include numbers with an unbounded number of prime factors using ideas from the Chinese remainder theorem. We know,

$X \equiv a_i \bmod p_i^{\alpha_i}$ $\quad (1 \le i \le N)$ where the p_i are distinct primes and the α_i are natural numbers, has a solution in the natural numbers.

We construct entities so that the conditions above are satisfied for all prime numbers.

We consider matrices (a_{ij}) where a_{ij} satisfies $0 \le a_{ij} < p_i^j$. To ensure consistency with what happens with natural numbers we will insist that $a_{ij} \equiv a_{i\,j+1} \bmod p_i^j$ always.

Thus the i^{th} row of the matrix specifies the residues for the powers of the prime p_i.

Then the matrix meaning of $(a_{ij}) \equiv a_{ij} \mod p_i^j$ is defined by $a_{ij} \equiv a_i \mod p_i^j$.

Each matrix then defines a quantity and the natural numbers are included.

Addition and multiplication of the matrices are just position wise addition and multiplication (with reduction modulo p_i^j) and it is easy to verify that the consistency condition is retained. The new system is closed under addition and multiplication. This matrix construction then allows the extension of the Chinese remainder to the case of all prime numbers. Clearly, the natural numbers may be imbedded in this collection.

Next, we consider the equivalence relation A~B iff A = UB where U is a unit in the structure.

This collection of equivalence classes E, with addition and multiplication defined is then regarded as the extension of the natural numbers in that it includes entities divisible by an unbounded number of prime powers.

We note that XY = 0 is possible with non- zero X and Y but this does not happen below.

Then in this domain, the expansion of $\Pi(1+p+p^2+p^3+ ....+..)$ includes all finite natural numbers and all unbounded natural numbers even though no new primes are introduced. The cardinality of these numbers is the cardinality of the continuum rather than the cardinality of the natural numbers which is implicit in $\zeta(s) = \Pi(1+p^{(-s)}+p^{(-2s)}+p^{(-3s)}+ ....+..) = \sum 1/n^s$.

Note that in the elements of E if we just consider prime divisors for each prime we have the choice N ≡ 0 mod p or N ≡ 1 mod p.

These numbers alone account for $2^{|N|}$ elements and this is the cardinality of the real numbers.

Bibliography

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[2] Titchmarsh E. C. The theory of the Riemann zeta function, Oxford University Press, 1951.