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A discussion about the unprovability of the Riemann Hypothesis (RH) –

Peter Braun

This first part of this paper has been extended by 'An elementary argument about the Riemann hypothesis' which follows on from this part. The early section contains the mathematical results used in the later discussion and also backgrounds the thinking of the later framework. The intention is to iterate towards a formal explanation. The later part of the paper includes a tangible definition for 'essentially different theorems ' and a semi- formal discussion of the structure necessary to support the arguments.

Conventions:

All integrals are 1 to ∞.

All Dirichlet series sums are 1 to ∞.

All summations of simple number theoretic functions are over natural numbers $\leq$ x.

Wherever ϵ appears in text it is assumed to be any arbitrarily small positive real number.

Let F be a real valued function defined for positive real numbers. $F(X) = \Omega_{+-}(X^a)$ as $X \to \infty$ means the existence of positive numbers a, b and increasing sequences of positive natural numbers $\{x_i\}$, $\{y_i\}$ where $\lim x_i = \lim y_i = \infty$ as $i \to \infty$ for which $F(x_i) > ax_i^a$ and $F(y_i) < -by_i^a$ for i = 1,2,3 .... This is the extended Landau omega notation as normally used in number theory.

Oscillatory results for summation functions related to the Möbius and Liouville functions are given in Braun [1]. At the time only fragmented results connecting the oscillatory behaviour of certain number theoretic functions and the placement of the non-trivial zeros of the zeta function were found in the literature.

Two theorems which are assumed:

1. Let $f(s) = \sum a(n)/n^s$ where $A(x) = \sum a(n)$. Then $f(s) = \int A(x)/x^{(s+1)}\,dx$ (the integral representation for Dirichlet series), Titchmarsh [1].
2. If the a(n) are real and eventually of one sign then the function represented by the series has a singularity at the real point on the line of convergence of the series, Titchmarsh [2].

Let $M_1(x) = \sum\mu(n)$ and for K > 1, $M_K(x) = \sum M_{(K-1)}(x)$ where μ is the Möbius function

Theorem 1

For $K \geq 1$ let $L_K(s) = \sum M_K(n)/n^s$.

$L_K(s) = 1/(s-1)(s-2)\ldots(s-K)\zeta(s-K) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Proof

We note the trivial estimate $M_K(x) = O(x^K)$ as $x \to\infty$.

The proof is by induction. The method may be used to verify the theorem for K= 1.

$$L_{(K+1)}(s) = \sum M_{(K+1)(}n)/n^s = s \int \{\sum M_{(K+1)(}n)\}/x^{s+1}\,dx$$
$$= s \int\{\sum[x]\text{-}n+1)M_K(n)\}/x^{s+1}\,dx$$

$= s \int \sum M_K(n)/ x^s dx - s \int \sum nM_K(n)/ x^{s+1}dx + p_K(s)$ where $p_K(s)$ is analytic for $\sigma > K+1$.

Thus $L_{(K+1)}(s) = \{(s/(s-1))L_K(s-1)\} - L_K(s-1) + p_K(s)$

$= \{1/(s-1)\} L_K(s-1) + p_K(s)$ and the result follows.

We use this result to derive the main results about the oscillatory behaviour of the Möbius sum function and the higher summation functions.

Let $\underline{\sigma}$ be the smallest real number such that $\zeta(s) \neq 0$ for $\sigma > \underline{\sigma}$.

Theorem 2

$\mathbf{M}_K(x) = \Omega_{+-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to \infty$.

Proof

Suppose $M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)}$ is eventually of one sign, where A is a non-zero integer.

Then the function defined by the Dirichlet series, $H_K(s) = \sum(M_K(n)+ An^{(K-1+\underline{\sigma}-\epsilon)})/n^s$ has a singularity at the real point on its line of convergence.
It follows from the preceding theorem that
$H_K(s) = 1/[(s-1)(s-2)\ldots(s-K)\zeta(s-K)] + A\zeta(s-K+1-\underline{\sigma}+\epsilon) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Moving from right to left along the real axis we find the first singularity of $H_K(s)$ at $s-K+1-\underline{\sigma}+\epsilon = 1$.
i.e $\sigma = K+\underline{\sigma}-\epsilon$.

Since $H_K(s)$ is then analytic for $\sigma > K+\underline{\sigma}-\epsilon$ it follows that $\zeta(s-K)$ is analytic for $\sigma > K+\underline{\sigma}-\epsilon$.
In other words $\zeta(s)$ is analytic for $\sigma > \underline{\sigma}-\epsilon$. This contradicts the choice of $\underline{\sigma}$.

Application to the Riemann hypothesis

In earlier explanations, Braun [1], [2], [3] and [4] a principle was discussed which deals with finite proof and unprovability. The principle was coined the 'finite proof, finite theorem assertion' (FPFTA) and the simplest form is that

'A theorem which admits a finite proof will only generate a finite number of essentially distinct **theorems'.**

Mathematicians are not averse to thinking in pictures or using imagery to help understand what is being talked about. Some props which may help to visualise FPFTA are listed below:

- As an accounting or counting exercise; we cannot get more out of a theorem than the quantities which define the theorem (and these are finite and bounded). We may see this as an application of Newton's third law
- A finite argument only contains a finite number of different components which need relating in some way. If we identify a theorem with its proof, since the proof will only generate a bounded number of logically different theorems (allowing for inductive collapsing), so too will the original theorem
- An analogy with linearly independent vectors in a vector space

- Picture the totality of everything as finite – rather large – but still finite. Realise that unbounded is made up via a pattern like 1, 1+1, 1+1+1 …. but verifiable argument must necessarily be finite. The only way of generating an unbounded number of theorems is through an inductive mechanism of some sort and these theorems are then in a special logical relationship with each other held together by the acceptance of the unbounded pattern 1, 1+1, 1+1+1…..

The reader is invited at this point to avoid prematurely looking for a counter example or to disentangle the sentence to uncover a tautology or self- fulfilling prophesy or some such thing. Further explanation should provide context. A prior condition we need is to clearly understand what we mean by a truth in this context. We take the view here that truth is derived by exhausting finite possibilities from other things that we take to be true.
For example, consider objects A and B. We take as true that exactly one of these objects is contained in a certain jar. Given the information that B is not in the jar, we say it is true that A is in the jar.
Even though we may consider this an inductive argument we also take it as true that no violation of this will occur in experiments. It will always be verified by experiment. And this is the extent of mathematical truth: a belief that the jar experiment will never lead to contradiction. This belief has the same hierarchy as modus ponens. It is the thinnest sort of thing you can have without having nothing – and they are called logical laws. Now with Cantor's diagonal argument in **set** theory we can no longer depend on finite truth. The jar is supposed to contain an unbounded number of items. To reach a conclusion in the argument we **assume** the truth of an unbounded argument. This produces a branch point in mathematical thinking around the cardinality of the real numbers. In the Cantor diagonalisation proof of uncountability Wikipedia [1], we have the set of all unbounded 0 , 1 sequences $\{s_n\}$ and then a 0,1 sequence which is not a member of this set. And this unbounded argument is taken to conclude the reals are uncountable. When Bertrand Russell uses a similar argument it is regarded as a paradox or antimony. The unbounded argument is $s_0 \neq s_1$, $s_0 \neq s_2$, $s_0 \neq s_3$ ….. In the finite universe, only bounded arguments are accepted because they are defensible by classical logic. Then in Peano arithmetic, taking place in a finite universe, the continuum hypothesis is unprovable – the logic runs out. It is important to recognise here the branching – the acceptance of the unbounded argument as true. Cohen [1] succeeded in 1963 in establishing the independence of the continuum from Zermelo-Fraenkel set theory in a formal argument. On the face of it there is a need for an axiom which says that R is a set. The fact that Cohen's proof was formally recognised in mathematics with a Fields prize indicates a new bridge between mathematics and logic.

In the finite universe, valid arguments will look like p_1^p_2^…p_n where n is a counting number but Cantor set theory allows a quasi-logical argument p_1^p_2^…p_n ……, relying on a quasi-truth by insisting on a jam jar image of the infinite set. Fortunately, there is no crisis to attend to because all the constructions in analysis which lead to classical complex analysis may be carried out without the need for this sort of thinking and imagery. However, any difficulty in seeing the FPFTA as anything other than trivial may be caused by a sub-conscious conditioned belief that unbounded arguments are verifiable. They are not. A comparison of sorts is perhaps in literature in the difference between true stories and fiction. It doesn't quite make sense to call a work of fiction true but it does not take away the validity of the form by calling it fiction. There is then no question about the compatibility of Cantor set theory as it is developed in classical complex variable theory and the arithmetic which can be developed from the finite world of arithmetic. If a result is proven in the Cantor realm it cannot be contradicted in the arithmetical realm, but if a theorem is provable in the Cantor realm but is unprovable in the arithmetical realm it means that numerical calculation will never conflict with the result.
We next discuss theorems which assume in some way the modified axiom of infinity. That is, somewhere in the theorem, an analysis will uncover the assumption that the pattern [1], [1+1], [1+1+1], [1+1+1+1] … may be continued indefinitely. This is not a theorem to prove but is an

assumption to be accepted. This does seem like a reasonable assumption, but we are excluding the existence of some reason why this acceptance may be questioned. In this notion of unbounded we are not looking to extend finite argument, only recognition of indefinite extension of pattern – the extension is however always finite. We can thus make a distinction between finite mathematics (FM) and mathematics in which this assumption of unboundedness is clearly present, and we call the non- finite mathematics 'unbounded mathematics, (UM). In this discussion it is not important to obtain definitions for FM and UM which are mutually distinct classes. In fact the basic idea of proof of theorems in UM is the mechanism: (f Є UM) + proof → (f Є FM). The idea is that with f Є FM we have an unbounded collection of elements or theorems {p(1), p(2), p(3) ... } and an unbounded number of different things to check in the collection before we are able to announce something or other as true. A finite proof consists of finding enough patterns, in the theorems in the set, to reduce the verification to a finite exercise. This allows a process of applying the rules and assumptions and getting to a point (finitely) where there is nothing left to prove. FPFTA is about the distinction between sets of theorems for which this is possible and sets of theorems for which it is not possible. Unprovability is about proving something is not there – a finite proof - and this is going to be a different sort of proof.

In terms of 'global' equations we may express the other side of (f Є UM) + proof → (f Є FM), using the comparisons [(f Є UM) + (bounded pattern) → proof] and [(f Є UM) + (unbounded hierarchy) → unprovability]. The application of FPFTA to an unprovability proof needs to demonstrate sufficient unbounded hierarchy to get a proof. We look at some examples before moving to RH.

Non examples and examples

1.Let q(N) denote the sum of the first N natural numbers. Let p(N) denote the theorem

q(N) =(1/2)N(N+1). The collection {p(1), p(2)....} is then a true theorem.

Note we move away from calling q(1)^q(2)^q(3) ....' the theorem' because this collection of hieroglyphics does not have a meaning and if we are thinking in terms of the logical connective 'and' which only has sensibility in the finite case, to attach a meaning to lim(q(1) ^q(2)^q(3) ....^q(n)) as n→∞, retaining some meaning for 'and', just leads to difficulties. Classical logic is about finite argument and there is little point in trying to get an extension to the unbounded case as we cannot observe the unbounded case. We may only verify the unbounded case if there are inductive mechanisms which reduce the verification to a finite number of cases. The axiom of infinity is about unbounded pattern rather than 'infinite logic'. We see for example, convergent series have finite meaning through finite logic. We need to keep the logic and constructions of number theory clearly unbundled to avoid confusion. In constructions through to the complex numbers there is no extra quantity of entities created in the sets beyond the initial assumption of unbounded in the axiom of infinity. The hierarchy of different sized infinite sets is theory derived in set theory about sets. It creates an extra 'logic' to go with the visualisation of unbounded sets in set theory. Mathematicians know how effective this approach has been, but the admission of the unbounded argument obscures the use of finite arguments using classical logic to recognise unprovability. The classical logic we use to develop complex analysis uses a notion of availability of entities 'for any chosen $\epsilon > 0$' rather than imagining **all** the members of an unbounded set which is never going to be possible in a finite universe. We just need to know there will not be an exception to 'for any chosen $\epsilon > 0$'.

The use of tidy logical notation involving the universal quantifier does not add any additional legitimacy to a theorem, in terms of mathematical sensibility, and would seem to have more to do with fashion than anything – a style of presenting argument. In other words saying 'for all' does not get beyond the finite in terms of what is verifiable but merely acknowledges the acceptance of unbounded pattern.

2. A more intricate example was suggested by King [1] offering a 'devil's advocate' position concerning FPFTA. Namely, the Wiles proof of Fermat's last theorem (FLT) which we assume to be proven in the conventional mathematical sense. The considerations of this example and the next example highlighted the need to clarify the explanation of FPFTA to bring it up to a workable principle. Let p(n) be the statement that $X^n + Y^n = Z^n$ does not have non-trivial solutions. FLT is essentially the assertion that the unbounded collection of theorems {p(3), p(4) .....} are all true. On the face of it, we may think we are looking at a candidate theorem for FPFTA. In the pre-proof days, the known relationships between theorems in the collection were fragmented and although the problem was reduced to such things as n prime and a non-regular prime, the pattern required for finite proof was missing. The location of sufficient pattern to provide reduction to a finite proof provided a basis for the proof to be accepted. This necessarily involved inductive mechanisms, albeit very complicated, which found commonality in the theorems in the set.

3. This example involves the Riemann zeta function and led to a clarification of FPFTA and an extension.

Let $M_1(X) = \sum\mu(n)$ $(1 \leq n \leq X)$ where μ is the Möbius function and let $M_K(X) = \sum M_{(K-1)}(n)$ $(1 \leq n \leq X)$, $K > 1$.

A proof that $M_K(X) = \Omega_{+-}(X^{K-(1/2)-\epsilon})$ as $X\to\infty$ assuming RH is included at the beginning of this discussion. If we let p(K) denote the theorem $M_K(X) = \Omega_{+-}(X^{K-(1/2)-\epsilon})$ as $X\to\infty$ then the set {p(1), p(2)....} is a true theorem assuming RH. The theorems are logically different in the sense that more is being asked of $M_K(X)$ in the amplitude of the sign oscillation than for $M_{K-1}(X)$ (we easily construct examples where $A_{K-1}(X)$ has this oscillatory property but, through a dampening in the averaging, $A_K(X)$ does not have the corresponding level of oscillation). RH then generates an unbounded number of logically connected but different theorems, theorems which have different strength. How then is this different from the second example of the Wiles proof of Fermat's last theorem (WFLT)?

With WFLT there is no suggestion that there is an algebraic/ logical hierarchy in the theorems p(3), p(4)….. . For example if we had p(n) true → p(n+1) true, but not the reverse implication we could create a strength hierarchy and start thinking about FPFTA. A more profound explanation links the theorems p(3), p(4)… through Wiles theorem/proof. Here, there is no FPFTA for Fermat's last theorem because there is a common inductive mechanism which puts all the p(n) on an equivalent logical plane.

Returning to the FPFTA of RH, consider the problems in arithmetic of proving an Ω_{+-} theorem for the Möbius sum function. Clearly, in the conventional weighting of things in number theory, proving $M_1(X) = \Omega_{+-}(X^a)$ as $X\to\infty$ is a stronger result than proving $M_1(X) = \Omega_{+-}(X^b)$ if $a > b$. In this sense RH is the weakest possible result. On the other hand (assuming we know there exists a line $\sigma = \underline{\sigma}$ ($\underline{\sigma} < 1$) such that RH($\underline{\sigma}$) is true), and we focus on proving RH(a) or RH(b) true, then proving RH(a) true is a weaker result than proving RH(b) with $a > b > \underline{\sigma}$.

Each result p(1),p(2) .. is asking more and more of the Möbius function in terms of logical hierarchy. If we use FPFTA the solution set is unprovable. Although this is sufficient for the author to conclude the unprovability of RH, the essential testing process described in van der Poorten [1] has produced an extension of FPFTA to support the undecidability of RH($\underline{\sigma}$) for $\frac{1}{2} \leq \sigma \leq 1$.

A continuum of undecidables

For convenience we let FPFTA(½) denote the theorem in the preceding section. We have proved that $M_K(X) = \Omega_{+-}(X^{K-(1-\underline{\sigma})-\epsilon})$ as $X\to\infty$ for $K \geq 1$, follows from RH($\underline{\sigma}$) in Theorem 2. We thus have a continuum of unbounded theorem sets of the FPFTA form with FPFTA($\underline{\sigma}$) following from RH($\underline{\sigma}$)).

The same curious reversal of theorem strength as discussed in section 2, occurs here if we assume the value of $\underline{\sigma}$ is decidable in [1/2, 1).
Suppose the value of $\underline{\sigma}$ is a decidable theorem and the value of σ is $\underline{\sigma}$ with $\underline{\sigma} < 1$.Then moving to decreasing numbers σ less than 1 towards $\underline{\sigma}$, we find RH(σ) is a theorem increasing in strength but FPFTA(σ) is a theorem in Peano arithmetic of decreasing strength. We suggest that this contradictory state disallows RH($\underline{\sigma}$) as a decidable theorem. We note RH(1), which we have not considered, may be thought the strongest possible provable theorem. $\zeta(s)$ would then have zeros arbitrarily close to $\sigma = 1$. Since FPFTA($\underline{\sigma}$) is unprovable for any $1 > \underline{\sigma} \geq \frac{1}{2}$ we will not find zeros arbitrarily close to $\sigma = 1$.

We have noted in Braun [3] that unprovability means all non-trivial zeros of $\zeta(s)$ through computation will lie on $\sigma = \frac{1}{2}$ and are simple zeros.

Notes

1. Try to prove all of $\Pi(x) = x/\ln(x) + o(x/\ln(x))$, $\Pi(x) = x/\ln(x) + x/\ln^2(x) + o(x/\ln^2(x))$, $\Pi(x) = x/\ln(x) + x/\ln^2(x) + 2x/\ln^3(x) + o(x/\ln^3(x))$, ….as $x \to \infty$ in Peano arithmetic. Since this is not possible numerical investigation will never contradict RH.
2. A short line of thought assuming FPFTA is as follows:
   Since RH is undecidable in Peano (finite) arithmetic, the only possible 'proofs' will involve unbounded logical argument of the Cantor variety. A numerical investigation (with associated theory) into the placement of the zeros is essentially within Peano arithmetic. Consequentially, since an exceptional zero would be a (dis) proof of RH there will not be any numerical evidence to disprove the hypothesis. Thus all the zeros encountered will be on the line $\sigma = \frac{1}{2}$ and all will be simple.
3. The convergence of each of $\{\{1-2/2^s\}\zeta(s)\}^k$ in $\sigma > 1/2$ assuming the Lindelöf hypothesis (LH) indicates that this hypothesis may be unprovable using the FPFTA principle. This unprovability would imply the unprovability of RH. It would be curious to think of LH as a stronger theorem than RH.
4. Again, if we focus on the Ω oscillatory properties of the $M_K(x)$ in arithmetic and for lack of any better idea appearing, appeal to common sense: as we have noted, $M_K(x) = \Omega_{+-}(X^{a+K-1})$ (K = 1,2,3..) as a series of theorems is asking less than $M_K(x) = \Omega_{+-}(X^{b+K-1})$ (K = 1,2,3..) with a<b. Then the weakest result of all is the truth of the Riemann hypothesis.
5. Another interesting result about oscillatory behaviour driven pretty much by the same mechanism is the behaviour of $\varphi(x) - [x] = \{\sum\Lambda(n)\} - [x] = \sum a(n) = A_1(x)$, where as usual Λ denotes the von Mangoldt function. We have $-\{\zeta'(s)/\zeta(s)\} - \zeta(s) = \sum a(n)/n^s$ and with $A_K(x) = \sum A_{(K-1)}(n)$, the methods of theorem 1 and 2 allow $A_K(X) = \Omega_{+-}(X^{K-(1-\underline{\sigma})-\epsilon})$ as $X \to \infty$ for $K \geq 1$. This produces another FPFTA type example.
6. To get the sort of containment for this approach to get any recognition requires getting something like FPFTA beyond dispute. What is needed is a workable principle which is insulated from counter-example linked to verifiable truth in the sense discussed above. Otherwise, a lack of counter-example would not be regarded as sufficient.

Conventional causal proof

If one believes in the preceding discussion, the main thing discussed is that the Riemann hypothesis cannot be proven false in any extension of Peano arithmetic and this includes numerical investigation.

This does not exclude the possibility of a proof using unbounded argument in the Cantor realm where the truth discussed earlier is shaped into the extended type of truth, accepting the validity of the unbounded argument as an explanatory form. Those who want completeness to trump unprovability may be sceptical and disappointed that an explanation of RH and like theorems may lie in understanding mathematical thought rather than understanding mathematics.

It is pertinent to finish with a comment Sarnak [1] makes about Paul Cohen:-
'As mentioned above, it was his strong belief that such problems have simple solutions once properly understood'

In the paper following this discussion a definition of 'essentially different theorems' is provided which along with a picture of the argument hierarchies involved, exposes a simple explanation of RH type theorems.

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(continued)

The Riemann hypothesis in arithmetic

For the formal approach to elementary methods in analytical number theory, the article of Avigad [1] provides a good background for those, like the author, who are not particularly trained in the algebra of logical systems.

There is a fundamental issues identified in explaining the Riemann hypothesis which needs to be clarified. There is nothing here which is not well known but it is in the gathering together of certain observations in a particular order which supports the explanation.

The issue relates to working in a finite universe. This is assumed to be the universe as commonly understood: - what we think of as everything, as recognised by our senses and all the processes we experience as human beings and all the things we cannot exclude because we have not yet appeared as part of future discovery and experience. This is assumed to be finite but if there is objection to this we simply take an appropriate finite slice which cuts out the imponderables. A normal distinction of sorts is often made between the internal world of the individual and the external world of observation and action. This finite universe creates the restraint that at least in theory, all things which could be thought of as requiring an explanation before they are to be taken as true, are capable of true explanation. We do not assume an essential uniqueness of explanation although one may conjecture that this is so, but this may be a matter of policy rather than truth. In this discussion we talk about two types of mathematical truth and to distinguish between the type being talked about we use the words 'truth' and 'Truth'.

Truth will be taken to be that most solid of things – verification by finite exhaustion using other Truths:

In a simple case – a jar is known to contain either a red ball or a black ball. With the information that it does not contain the red ball, we take it as True that it contains the black ball. It is expected outcome and if an experiment produces a counter example we call it a trick or some such thing. We thus have a notion of Truth and defend it through language and experience.

Theorems in finite arithmetic are accepted if the explanation is reducible to this type of Truth.

We do not need axioms or set theory or a formal development to sure up the subjective belief in the validity of these arguments – they are as good as they possibly in terms of what we mean by Truth. The process of reaching a consensus view in an explanation is helped by using agreed on language and agreed on concepts which have precise meanings and using rules which are agreed on. A lot of mathematics proceeds along these lines at a functioning level rather than a completely formal level. If we have a couple of theorems P and Q with Q following as a Truth from the Truth of P we can go backwards and examine the assumptions and rules which have led to P and which have led Q. The orientation here is that P is at least as strong as Q and stronger than Q if either P follows from Q is false or requires a wider number of assumptions from those proving Q follows from P.

True explanations use earlier Truths as building blocks and use Truth as the method of arriving at new Truths.

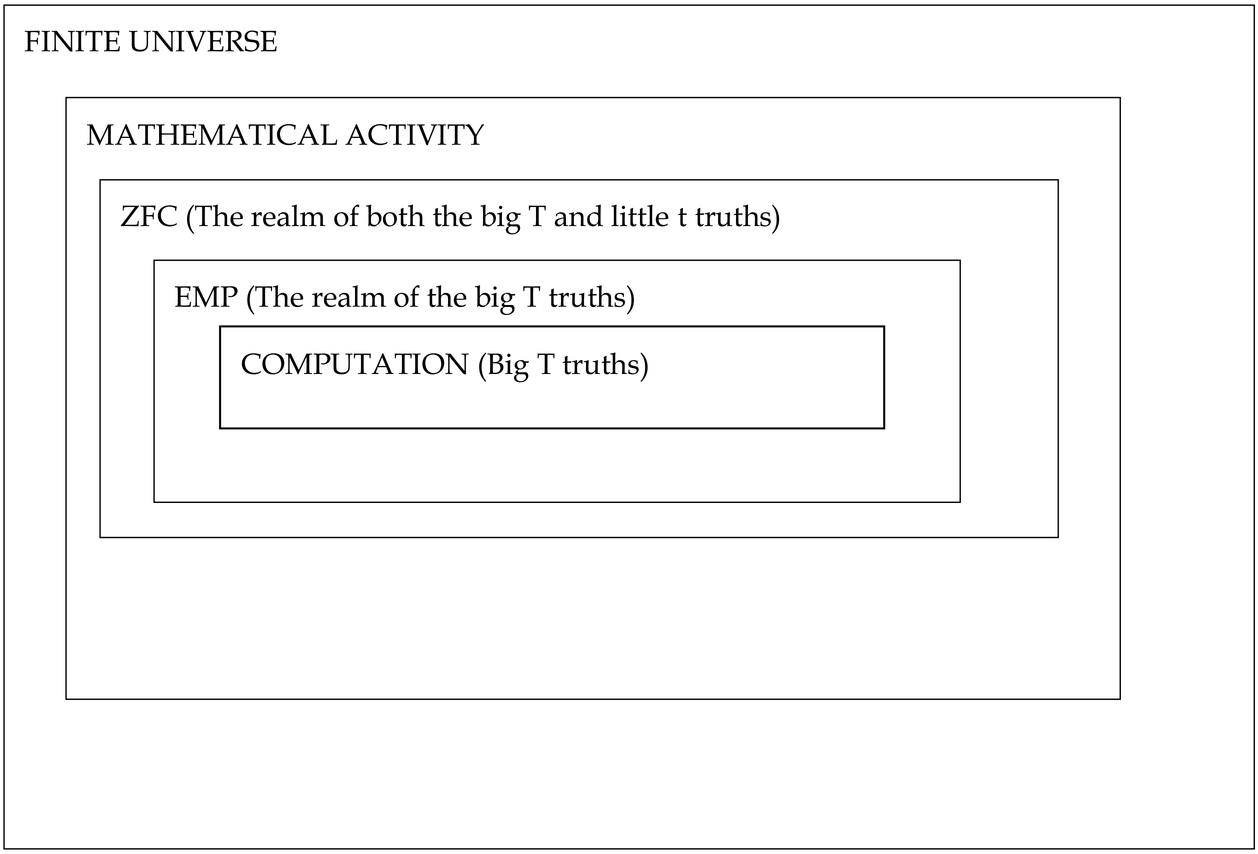
The other truth of which we need to be aware is the constructed truths found in such things as ZF and ZFC and some of the arguments in Cantor set theory. These truths are used in the same way as Truths. They are building blocks using Truths and truths to construct theorems which may then be taken as true or conditionally True. In some cases the theorems depend on special rules or axioms which are not verifiable by the proof by finite exhaustion method. Use of the axiom of infinity and particularly the axiom of choice are the key assumptions which sometimes lead to true or conditionally True theorems. This is a branching point in mathematics and a major development which has led to abstract mathematics. Roughly speaking, if an element of unboundedness is intrinsically linked with the proof of a theorem it is at best a true theorem or conditionally True theorem and sits both in the world of art and science. It is in art because it contains notions from the imagination which cannot be verified as True and it is in science because the results can never be contradicted by Truth constructions. To this extent ZFC is a consistent extension of arithmetic. ZFC and arithmetic are then the best possible friends. We do see though the potential for a lot of misunderstanding because mathematical theorems tend to be proved by mathematicians without too much concern for the Truth/truth mix and assuming that the logical profile of the theorem will be explainable by logicians. It may just be that at the interface of EMP and ZFC, which is normally the preserve of the logicians, there may be a fascinating boundary – a cease fire zone where there is much to understand. Through the boundary into logic land the inhabitants who also regard themselves as mathematicians, carry on like mathematicians, building Truths and conditional truths and looking at inter-relationships but in the algebra of logical systems.

The approach here is to see the Riemann hypothesis as a practical problem.

What we want as outcome is a True statement that in numerical investigation, the zeros of the zeta function will be on the line $\sigma = \frac{1}{2}$ and will be simple zeros. This is more satisfactory than an abstract conditionally true proof of RH, although such a thing cannot be excluded from the outcomes of this discussion.

The one piece of hand waving to be introduced is what we mean by arithmetic. This is not seen as something which needs to justified in a formal system. It is the real activity of doing arithmetic, bounded by the finite universe and explained in a formal sense in the finite axioms of Peano and other describing ZF axioms which mirror the dynamics of human thought in arithmetic, in machine like language. We label this system of activity EMP because it corresponds to the authors understanding of elementary methods in Peano arithmetic. The only possible proofs in EMP are bounded finite proofs. Note that
$\sum n = \frac{1}{2}N(N+1)$ is the theorem and not p(1)^p(2)^ ….

We now form a picture about the order of things with the usual Venn diagram interpretation.



The reader should note that the picture is not to scale except for those focussing on this sort of problem. To recap, we have:-

- An internal/external world of mathematical thought and activity in which theorems may be proven to be True with verification using the method of finite exhaustion (EMP)
- The finite universe has computational capacity and only uses the zero-one type logic in machine calculations, corresponding to logical thought in EMP (COMPUTATION)
- In this paragraph we drop the usage truth/Truth. Outside of EMP and enclosing it there is a realm which uses sets and axioms which we call ZFC. Some of the theorems in ZFC use a special type of argument and some theorems depend on this special type of argument. The truth of proved theorems is a different sort of truth since finite exhaustion may no longer enough to arrive at the validity of the mathematical proof. It assumes the unbounded argument which produces such things as Cantor's diagonalisation argument about the non-countability of the real numbers. The validity of the truth that a proof has been constructed however is still a finite truth within the rules of ZFC and the notion of finite truth. Some theorems in ZFC depend on unbounded argument and there is no verification in finite truth argument which can ever verify the truth or falsity of theorems of this type. Finally, no valid theorem in ZFC can be contradicted in EMP. We say ZFC and EMP are compatible or ZFC is a compatible extension of EMP. To repeat, within the mathematics of ZFC the true arguments are proven true through finite exhaustion but the mathematical truth is a different truth sometimes if it needs the axiom of infinity or the axiom of logical choice and these are not bounded finite notions.

In EMP we formulate an observation about theorems which have interpretation in EMP:

A provable theorem in EMP can only generate a finite number of essentially different theorems which are consequences of the theorem.

We define 'essentially different' as follows:

Firstly we consider propositions or theorems P_1, P_2 ....P_n which reference a well- defined class of objects in EMP.

P_1, P_2 ....P_n are theorems in ZFC (proven or unproven) which have interpretation in EMP

By $P_i \rightarrow P_j$ we mean the truth of P_j follows from the truth of P_i in EMP

The theorems $\{P_1, P_2 .... P_n\}$ are called essentially different if for each i j with $i \neq j$ at most one of $P_i \rightarrow P_j$ and $P_j \rightarrow P_i$ is provable for the class of objects in EMP.

If a theorem generates an unbounded number of essentially different theorems in EMP then we may construct an unbounded theorem in EMP which cannot be described using a finite set of rules. Hence, the theorem cannot be provable in EMP.

The generation of a contradiction in EMP through a proof in EMP would be against the compatibility of ZFC with EMP. Since computation is essentially an activity 'within' EMP computation evidence will neither prove nor disprove the theorem.

In the case of the Riemann hypothesis- with FPFTA: Let $M_1(X) = \sum\mu(n)$ and $M_K(X) = \sum M_{(K-1)}(n)$, summation $1 \leq n \leq X$, where μ is the Möbius function. We have noted $M_K(x) = \Omega_{+-}(x^{(K-1+\underline{\sigma}-\epsilon)})$ as $x \rightarrow \infty$ are theorems which follow from RH, they are essentially different and they have interpretation in EMP and form members of an FPFTA suite.Thus, RH is unprovable in EMP and so computed zeros can never prove or disprove RH. $\underline{\sigma}$ is the greatest number for which $\zeta(s) \neq 0$ for $\sigma > \underline{\sigma}$. Also, as a consequence, $M(x) = O(x^{½})$ as $x \rightarrow \infty$ in unprovable in EMP. Therefore all computed zeros will be simple. Finally, we also have the partial result in ZFC that RH cannot be proven false in ZFC.

Questions:

Q. In verifying the computed zeros we need to reference some ZFC~EMP 'unbounded' type theory. How do you respond to this?

A. Try telling your computer it is infinite. ZFC is still part of a finite universe. It's just a language with more words than EMP. Someone verifying the placement of the zeros using some theory is only ever going through finite logical processes. If you don't like this, note that ZFC is compatible with EMP so verification of the zeros has interpretation in EMP.

Q. OK but why can't we use the FPFTA in ZFC and say RH is unprovable full stop.

A. As noted, the concept of truth may change moving from EMP to ZFC. The unbounded part of the realm is imaginary and the results which involve 'unbounded' in an intrinsic way are not results which can be verified by computation. The thing is that ZFC results are always consistent with EMP and so true results will never be contradicted by computation if the results have that sort of amenable form.

These are true results in ZFC but cannot be proved in EMP. We would expect computation to be compatible with this without having to stretch our imaginations because of the expected compatibility of ZFC and EMP, but it is logically possible that the computed evidence would never look great. As the evidence is not a proof it wouldn't matter anyway.

Q. Back to hand waving, you've made up EMP but you haven't clearly defined it. Aren't you just invoking mystical properties of EMP which will allow the conclusion?

A. I'm glad you asked that. EMP is not a mystical realm though. It comes from counting and that sort of stuff – it is finite and only accepts truth and proof that is verifiable. Now we cannot prove that this is not subjective. What we can be happy with is the belief that the classical laws of logic underpin this stuff and they seem to be reliable. We just need to understand that all natural numbers are finite. We can get a semi -formal definition of EMP as the realm of big T truths in number theory. The other thing needed is that the results of computer investigations are big T truths.

Notes:

Finding a definition for essentially different propositions and getting a better handle on the status of the realms involved in a semi -formal way exposes the earlier drafts more clearly as early drafts. The most dramatic consequence is the interplay between ZFC and EMP. We see it remains logically possible that there is both a true proof and a True proof of RH. In the reverse direction, this may also be the case with Fermat's last theorem (FLT). The assumption in an earlier note of a deep inductive mechanism which places the p(n) statements on an equal logical footing is no longer necessary. It seems possible that the p(n) of FLT are essentially different theorems in EMP so we could not use the unprovability argument in EMP for FLT. The lost theorem in EMP remains a possibility, consistent with the conjecture of Harvey Friedman as described in Avigad [1]. Similarly, RH may have an abstract proof in ZFC.

As noted in another discussion in Braun [1] within ZFC we have the unconditional result that $M_K(x) = \Omega_{+-}(x^{(K-1/2-\epsilon)})$ as $x \to\infty$ for $K \geq 1$. This too is an FPFTA suite in EMP and is consequently not provable in EMP. That is , we need the full muscle of ZFC in the theory of the Riemann zeta function to derive this result.

References

[1] Jeremy Avigad

Number theory and elementary arithmetic. Philosophia Mathematica (3) Vol. 11 (2003), pp.257-284

[1] Peter Braun

Various discussion papers up to 2011:- https://www.peterbraun.com.au