Oscillatory behaviour of the Möbius function

Peter Braun

Notation and usage:

We extend the Landau notation o, O, Ω+,- to |o, |O, |Ω+,- where we wish to indicate the precise nature (either assumed or already proven) about the behaviour of a function within the limitations of these notions.

For example: let μ denote the Möbius function and let $M(x) = \sum \mu(n) \quad (1 \le n \le x)$. If the extended Merten's conjecture is true and $M(x) = O(x^{½})$ as $x \to \infty$ then we have $|O(x^{½})$ as $x \to \infty$. On the other hand, if the extended Merten's conjecture is false but the Riemann hypothesis is true then we have both and $M(x) = |O(x^{½+\epsilon})$ and $M(x) = |o(x^{½+\epsilon})$ as $x \to \infty$.

Generally, if $A(x) = \sum a(n) \quad (1 \le n \le x)$ then we also write $A_1(x) = A(x)$ and $A_k(x) = \sum A_{(k-1)}(n)$ $(1 \le n \le x)$.

We call the $A_k(x)$ the higher summation functions. Although we have a starting function $A_1(x)$ this is somewhat relative as we could from another starting function have $A_1(x)$ as one of the higher summation functions.

Framework

The essential frame of mind here is that of the familiar and traditional 'working' mathematician who is happy to work through constructions for things like rational numbers, integers, real numbers, complex numbers, p-adic numbers and so on with no doubt that the whole process is thoroughly sensible and does not contain assumptions which can be turned in on the results to arrive at contradictions. Set theory is viewed more as convenience in language to express constructions and mathematical discussions in a convenient compressed manner and not some sacred basis which can be used in pure reason to arrive at some ultimate explanation for the activity involved in a proof. In other words cognition is accepted as the overriding component in acceptance of proof. We cannot get away from the recognition that assumptions are being made in any meaningful explanation in any context and that this limitation leads to inter-relationships between things rather than 'proof' without assumption.

Thus, we assume a distinction between Peano arithmetic without the axiom of choice and without the axiom of infinity and classical complex analysis which assumes both these axioms in the justification of this activity through ZF set theory. We are not delving here into formal logical systems but assume no valid conclusion in classical complex analysis can be contradicted in Peano arithmetic.

Two theorems which are assumed in the following discussion:

1. Let $f(s) = \sum a(n)/n^s$ where $A(x) = \sum a(n)$. Then $f(s) = \int A(x)/x^{(s+1)}\,dx$ (the integral representation for Dirichlet series), Titchmarsh [1].
2. If the $a(n)$ are real and eventually of one sign then the function represented by the series has a singularity at the real point on the line of convergence of the series, Titchmarsh [2].

Let $M_1(x) = \sum\mu(n)$ and for $K > 1$, $M_K(x) = \sum M_{(K-1)}(x)$ where μ is the Möbius function

Theorem 1

For $K \geq 1$ let $L_K(s) = \sum M_K(n)/n^s$.

$L_K(s) = 1/(s-1)(s-2)\ldots(s-K)\zeta(s-K) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Proof

We note the trivial estimate $M_K(x) = O(x^K)$ as $x \to\infty$.

The proof is by induction. The method may be used to verify the theorem for K= 1.

$L_{(K+1)}(s) = \sum M_{(K+1)(}n)/n^s = s \int \{\sum M_{(K+1)(}n)\}/x^{s+1}\,dx$

$= s \int\{\sum[x]-n+1)M_K(n)\}/x^{s+1}\,dx$

$= s \int \sum M_K(n)/ x^s\,dx - s \int \sum nM_K(n)/ x^{s+1} + p_K(s)$ where $p_K(s)$ is analytic for $\sigma > K+1$.

Thus $L_{(K+1)(}s) = \{(s/(s-1))L_K(s-1)\} - L_K(s-1) + p_K(s)$

$= \{1/(s-1)\}\, L_K(s-1) + p_K(s)$ and the result follows.

We use this result to derive the main results about the oscillatory behaviour of the Möbius sum function and the higher summation functions.

Let σ be the smallest real number such that $\zeta(s) \neq 0$ for $\sigma > \sigma$.

Theorem 2

$\mathbf{M}_K(x) = |\Omega_{+-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to\infty$.

Proof

Suppose $M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)}$ is eventually of one sign, where A is a non-zero integer.

Then the function defined by the Dirichlet series, $H_K(s) = \sum(M_K(n)+ An^{(K-1+\underline{\sigma}-\epsilon)})/n^s$ has a singularity at the real point on its line of convergence.
It follows from the preceding theorem that
$H_K(s) = 1/[(s-1)(s-2)\ldots(s-K)\zeta(s-K)] + A\zeta(s-K+1-\underline{\sigma} +\epsilon) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Moving from right to left along the real axis we find the first singularity of $H_K(s)$ at $s-K+1-\underline{\sigma} +\epsilon = 1$. i.e $\sigma = K+\underline{\sigma} -\epsilon$.

Since $H_K(s)$ is then analytic for $\sigma > K+\underline{\sigma}-\epsilon$ it follows that $\zeta(s-K)$ is analytic for $\sigma > K+\underline{\sigma} -\epsilon$.
In other words $\zeta(s)$ is analytic for $\sigma > \underline{\sigma} -\epsilon$. This contradicts the choice of $\underline{\sigma}$.

The fact that this is the best possible result follows from the definition of $\underline{\sigma}$ (finish).

Without qualification we thus have the results that $\mathbf{M}_K(x) = \Omega_{+-}(x^{K-1/2-\epsilon})$ as $x \to\infty$ for K=1,2 ....

We view this result as an unbounded sequence of results about the Möbius function.

July/August 2011

As such it may be interpreted as a theorem about a function μ from a class of number theoretic functions a which satisfy $|a(n)| \leq 1$ for each value of n.

Looking at propositions rather than theorems we let $P(a,K) = A_K(x) = \Omega_{+-}(x^{K-1/2-\epsilon})$ as $x \to \infty$ for the function a in the class of functions satisfying $|a(n)| \leq 1$.

Clearly, we do not have $P(a,K) \to P(a, K+1)$ because the sign change in $A_K(x)$ may be dampened in the next order of summation.

Thus the results about the Möbius function represent an unbounded number of essentially different propositions.

The finite proof finite theorem assertion (FPFTA) is that this phenomen can only be exhibited through an artificial gluing together of results p(n) with the assertion that p(1)^p(2)^ ... is 'a theorem' or by a mathematical theorem proved by induction in Peano arithmetic or by choosing essentially different theorems which are true over an entire class of functions.

Assuming the validity of FPFTA we see the results $\mathbf{M}_K(x) = \Omega_{+-}(x^{K-1/2-\epsilon})$ as $x \to \infty$ for K=1,2 ....are not provable in arithmetic.

From theorem 2 we also have that the value of $\underline{\sigma}$ is undecidable in arithmetic since a proof that $\sigma = \underline{\sigma}$ in arithmetic would establish the validity of theorem 2 in arithmetic, running up against FPFTA.

The Riemann hypothesis is that $\underline{\sigma} = ½$ and we assume then that this is undecidable in arithmetic.

The final argument needed is that the numerical calculation of a zero off $\sigma = ½$ would constitute a proof in arithmetic that the Riemann hypothesis was false.

To this end rather than invent analysis through the classical ϵ, δ method we rewrite the whole lot talking about ϵ- convergence rather than convergence.

The essential frame of mind here is that of the familiar and traditional 'working' mathematician who is happy to work through constructions for things like rational numbers, integers, real numbers, complex numbers, p-adic numbers and so on with no doubt that the whole process is thoroughly sensible and does not contain assumptions which can be turned in on the results to arrive at contradictions. Set theory is viewed more as convenience in language to express constructions and mathematical discussions in a convenient compressed manner and not some sacred basis which can be used in pure reason to arrive at some ultimate explanation for the activity involved in a proof. In other words cognition is accepted as the overriding component in acceptance of proof. We cannot get away from the recognition that assumptions are being made in any meaningful explanation in any context and that this limitation leads to inter-relationships between things rather than 'proof' without assumption.

Thus, we assume a distinction between Peano arithmetic without the axiom of choice and without the axiom of infinity and classical complex analysis which assumes both these axioms in the justification of this activity through ZF set theory. We are not delving here into formal logical systems but assume no valid conclusion in classical complex analysis can be contradicted in Peano arithmetic. In a formal argument we may encounter statements that such and such may be assumed for all $\epsilon > 0$. For practical purposes in computation we are just concerned with things matching up to any specified degree of accuracy. We then take the notion of ϵ -convergence through the definition (for example) that f(x) is ϵ- convergent to l as $x \to a$ if there exists δ $(=\delta(\epsilon))$ such that $|f(x)-l| < \epsilon$ whenever $|x-a| < \delta$.

A simple example is the calculation of $\sqrt{2}$ to a specified degree of accuracy.

A rational number t_n may be calculated such that $| (t_n)^2 - 2| < 1/n^2$ and there is theoretically no limit to calculation. We could talk in terms of ϵ- convergence (where we have convergence in the analytical sense) rather than convergence but it would seem a tedious task to develop classical analysis in this language. However, if we imagine it has been done, we no longer have to worry about the axiom of infinity. This axiom is a problem for set theory and logic but not one for practical mathematics. Then in the calculation of the ϵ non-trivial zeros of the Riemann zeta function which just happen to be the zeros of the zeta function in ZF classical analysis there is no need whatsoever to worry about the axiom of infinity. In other words – the practical calculation of zeros of the Riemann zeta function is essentially an activity in Peano arithmetic. We may construct all the things we need from Peano's axioms using a finitist approach of ϵ -convergence. The Riemann hypothesis then has an interpretation in this realm that all the non-trivial zeros of the Riemann zeta function found through numerical computation will *appear* to lie on $\sigma = ½$.

The logician knows that the theory which produces the criteria for deciding on the placement of the n^{th} zero for $\zeta(s)$ in a numerical investigation assumes the axiom of infinity and perhaps the axiom of choice. However, a programmer following a set of decision rules will only ever be working within finite classical logic. The mathematician who prescribes the decision rules will only ever be working within finite classical logic. So, all the verification activity is just finite arithmetical activity. The fact that there exists a theory which contains unprovable assumptions which is consistent with arithmetic does not endow the mathematician or programmer with any extra powers of thought – they are bound by classical finite logic. With this distinction between the theory which contains assumptions which could never be decided and the everyday activities involving classical logic we see there is really quite a difference between the prime numbers of arithmetic and what might be called the analytic primes of analytic number theory.

We may leave open at this stage the possibility of a proof of RH in abstract number theory (which cannot be a proof in arithmetic) yet to be constructed but we note that such a proof is not necessary for a proof of RH. The un-decidability of this proposition in arithmetic is sufficient.

References

[1] Titchmarsh. E.C. The Theory of Functions. Oxford University Press. 2nd edition 1979.

[2] Titchmarsh. E.C. The Theory of the Riemann Zeta-function. Oxford Science Publications, 2nd edition. 1986.

July/August 2011