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A further note on the Riemann hypothesis (ii) – Peter Braun

Abstract:

For 1> σ ≥ ½ let RH(α) denote the theorem: ζ(s) is non-zero for σ > α; We call RH(α) the quasi Riemann hypothesis.

An assumption in this discussion is that the axioms for arithmetic allow the construction of complex analysis with the complex numbers as a simple field extension of the real numbers. The axiom of infinity allows unbounded sets of theorems to be considered, but proof of unbounded collections must relate back to finiteness and must involve some inductive mechanism. The quasi Riemann hypothesis is seen as undecidable in ½ ≤ σ < 1 because two inductive mechanisms move in conflicting directions and an unbounded hierarchical string of theorems follows from the assumption of the quasi Riemann hypothesis for any α: 1 > α ≥ 1/2. RH may be interpreted as the weakest theorem in this context, which implies unbounded numerical verification that the zeros lie on σ = ½ and are simple.

Notation and usage

Wherever ϵ appears in text it is assumed to be any arbitrary real positive number greater than zero.

If F is a real valued function defined for positive real numbers then $F(X) = \Omega_{+-}(X^a)$ as $X \to \infty$ means the existence of positive numbers a, b and increasing sequences of positive natural numbers $\{x_i\}$, $\{y_i\}$ where lim x_i = limy $y_i = \infty$ as x→ ∞ for which $F(x_i) > ax_i^a$ and $F(y_i) < -by_i^a$ for i = 1,2,3 ….

The results in this discussion assume basic analytical techniques in the theory of the Riemann zeta function Titchmarsh [1], [2].

Section 1

Introduction

In earlier explanations, Braun [1], [2], [3], a principle was discussed which describes the negation of finite proof. The principle was coined 'the 'finite proof, finite theorem assertion' (FPFTA) and the simplest form is that

'A theorem which admits a finite proof will only every generate a finite number of essentially distinct **theorems'**

Mathematicians are not averse to thinking in pictures or using imagery to help understand what is being talked about. Some props which may help to visualise FPFTA are listed below:

- As an accounting or counting exercise; we cannot get more out of a theorem than the quantities which define the theorem (and these are finite and bounded). We may see this as an application of Newton's third law
- As a balloon containing a bounded number of seeds (the theorem); once the balloon is burst (the theorem proved) there are only a bounded number of seeds (theorems) to work with (if we want to start relating fallen seeds to other things on the ground we are moving away from the theorem into outside linkage).

- A finite argument only contains a finite number of different components which need relating in some way. If we identify a theorem with its proof, since the proof will only generate a bounded number of logically different theorems (allowing for inductive collapsing), so too will the original theorem

This method of discourse needs to be capped as language is strong enough without restraint to form unhelpful images.

The reader is invited at this point to avoid prematurely looking for a counter example or to dis-entangle the sentence to uncover a tautology or self- fulfilling prophesy or some such thing. Further explanation should provide context.

The devil is in what is meant by essentially distinct theorems. We discuss theorems which assume in some way the axiom of infinity. That is, somewhere in the theorem, an analysis will uncover the assumption that the pattern [1], [1+1], [1+1+1], [1+1+1+1] ... may be continued indefinitely. This is not a theorem to prove but is an axiom to be accepted as true. This axiom does seem like a reasonable assumption, but we are excluding the existence of some reason why this acceptance may be questioned. The distinction is between finite mathematics (FM) and mathematics in which this assumption of unboundedness is clearly present, and we call the non-finite mathematics 'unbounded mathematics, (UM). In this discussion it is not important to obtain definitions for FM and UM which are mutually distinct classes. In fact the basic idea of proof of theorems in UM is the mechanism:
($f \in$ UM) + proof → ($f \in$ FM). The idea is that with $f \in$ FM we have an unbounded collection of elements or theorems {p(1), p(2), p(3) ... } and an unbounded number of different things to check in the collection before we are able to announce something or other as true. A finite proof consists of finding enough patterns, in the theorems in the set, to reduce the verification to a finite exercise. This allows a process of applying the rules and assumptions and getting to a point (finitely) where there is nothing left to prove. FPFTA is about the distinction between sets of theorems for which this is possible and sets of theorems for which it is not possible. Unprovability is about proving something is not there – a proof - and this is going to be a different sort of proof.

In terms of 'global' equations we may express the other side of ($f \in$ UM) + proof → ($f \in$ FM), using the comparisons [($f \in$ UM) + (bounded pattern) → proof] and
[($f \in$ UM) + (unbounded hierarchy) → unprovability]. The application of FPFTA to an unprovability proof needs to demonstrate sufficient unbounded hierarchy to get a proof. We look at some examples before moving to RH.

Section 2: Examples

1.Let q(N) denote the sum of the first N natural numbers. Let p(N) denote the theorem

$q(N) = (1/2)N(N+1)$. The collection {p(1), p(2)....} is then a true theorem.

Note we move away from calling q(1)^q(2)^q(3) ....' the theorem' because this collection of hieroglyphics does not have a meaning and if we are thinking in terms of a logical connective 'and' which only has sensibility in the finite case, to attach a meaning to lim(q(1) ^q(2)^q(3) ....^q(n)) as $n \to \infty$, retaining some meaning for 'and', just leads to difficulties. Classical logic is about finite argument and there is little point in trying to get an extension to the unbounded case as we cannot observe the unbounded case. We may only verify the unbounded case if there are inductive mechanisms which reduce the verification to a finite number of cases. The axiom of infinity is about unbounded pattern rather than 'infinite logic'. We see for example convergent series have finite meaning through finite logic. We need to keep the logic and constructions of number theory clearly unbundled to avoid confusion. In constructions through to the complex numbers there is no extra quantity of entities created in the sets beyond the initial assumption of unbounded in the axiom of infinity. The hierarchy of different sized infinite sets is theory derived in set theory about sets. It does not create any extra logic. The classical logic we use to develop classical complex analysis uses a notion of availability of entities 'for any chosen $\epsilon > 0$' rather than imagining **all** the members of an unbounded set which is never going to be possible in a finite universe. We just need to know there will not be an exception to 'for any chosen $\epsilon > 0$'.

The use of tidy logical notation involving the universal quantifier does not add any additional legitimacy to the theorem and would seem to have more to do with potty training than anything – a style of presenting argument. In other words saying 'for all' does not get beyond the finite in terms of what is verifiable but merely acknowledges the acceptance of unbounded pattern.

2. A more intricate example was suggested by King [1] offering a 'devil's advocate' consideration concerning FPFTA. Namely, the Wiles proof of Fermat's last theorem (FLT) which we assume to be proven in the conventional mathematical sense. The considerations of this example and the next, highlighted the need to clarify the explanation of FPFTA to bring it up to a workable principle. Let p(n) be the statement that $X^n + Y^n = Z^n$ does not have non-trivial solutions. FLT is essentially the assertion that the unbounded collection of theorems {p(3), p(4) .....} are all true. On the face of it, we may think we are looking at a candidate theorem for FPFTA. In the pre-proof days, the known relationships between theorems in the collection were fragmented and although the problem was reduced to such things as n prime and a non-regular prime, the pattern required for finite proof was missing. The location of sufficient pattern to provide reduction to a finite proof provided a basis for the proof to be accepted. This necessarily involved inductive mechanisms, albeit very complicated, which found commonality in the theorems in the set.

3. This example involves the Riemann zeta function and led to a clarification of FPFTA and an extension.

Let $M_1(X) = \sum\mu(n)$ $(1 \leq n \leq X)$ where μ is the Möbius function and let $M_k(X) = \sum M_{(k-1)}(n)$ $(1 \leq n \leq X), k > 1$.

A proof that $M_k(X) = \Omega{+}{-}(X^{k-(1/2)-\epsilon})$ as $X\to\infty$ assuming RH is easily seen using well known ideas which were used in Braun [1] to regularise fragmented results about oscillatory behaviour of certain number theoretic functions. The methods are more easily accessible in Braun [2]. The pivotal result used is the classical result that if the integer coefficients of a Dirichlet series are eventually of one sign, the function defined by the series, has a singularity at the real point on its line of convergence Titchmarsh [2]. If we let p(k) denote the theorem $M_k(X) = \Omega{+}{-}(X^{k-(1/2)-\epsilon})$ as $X\to\infty$ then the set {p(1), p(2)....} is a true theorem assuming RH. The theorems are logically different in the sense that more is being asked of $M_k(X)$ in the amplitude of the sign oscillation than for $M_{k-1}(X)$ (we easily construct examples where $A_{k-1}(X)$ has this oscillatory property but, through a dampening in the averaging, $A_k(X)$ does not have the corresponding level of oscillation. We have generated an unbounded number of logically connected but different theorems. How then is this different from the second example of the Wiles proof of Fermat's last theorem (WFLT)?

With WFLT there is no suggestion that there is a algebraic logical hierarchy in the theorems p(3), p(4)….We don't have any general scheme. For example if we had p(n) true → p(n+1) true, but not the reverse implication we could create a strength hierarchy and start thinking about FPFTA. A more profound explanation links the theorems p(3), p(4)… through Wiles theorem/proof. Here, there is no FPFTA for Fermat's last theorem because there is a common inductive mechanism which puts all the p(n) on an equivalent logical plane.

Returning to the FPFTA of RH, consider the problems in arithmetic of proving an Ω_{+-} theorem for the Möbius sum function. Clearly, in the conventional weighting of things in number theory, proving $M_1(X) = \Omega_{+-}(X^a)$ as $X\to\infty$ is a stronger result than proving $M_1(X) = \Omega_{+-}(X^b)$ if $a > b$. In this sense RH is the weakest possible result. On the other hand (assuming we know there exists line $\sigma = \underline{\sigma}$ ($\underline{\sigma} < 1$) such that RH($\underline{\sigma}$) is true), and we focus on proving RH(a) or RH(b) true, then proving RH(a) true is a weaker result than proving RH(b) with $a > b > \underline{\sigma}$.

Each result p(1),p(2), ... is asking for an additional Möbius function property in terms of logical hierarchy and using FPFTA the solution set is unprovable. Although this is sufficient for the author to conclude the unprovability of RH, the essential testing process described in van der Poorten [1] has produced an extension to FPFTA which we label FPFTA(R), where R denotes to continuum. We add further strength to the argument.

Section 3

A continuum of undecidables

For convenience we let FPFTA(1/2) denote the theorem in the preceding section.

The techniques in Braun [1], [2] may be used to prove that $M_k(X) = \Omega_{+-}(X^{k-(1-\alpha)-\epsilon})$ as $X\to\infty$ for $k \geq 1$, follows from RH(α). We thus have a continuum of unbounded theorem sets of the FPFTA form (FPFTA(α) follows from RH(α)).

The same curious reversal of theorem strength as discussed in section 2, occurs here if we assume the value of α is decidable in [1/2, 1).
Suppose the value of α is a decidable theorem and the value of α is θ with $\theta < 1$.Then moving to decreasing numbers σ less than 1 towards θ, we find RH(σ) is a theorem increasing in strength but FPFTA(σ) is a theorem in Peano arithmetic of decreasing strength. We conclude that this contradictory state disallows RH(θ) as a decidable theorem. We note RH(1), which we have not considered, may be thought the strongest possible provable theorem. $\zeta(s)$ would then have zeros arbitrarily close to $\sigma = 1$. But then FPFTA(α) for any $1 > \alpha \geq \frac{1}{2}$ generates the required FPFTA.

We have noted in Braun [3] that unprovability means all non-trivial the zeros of $\zeta(s)$ through computation will lie on $\sigma = \frac{1}{2}$ and are simple zeros.

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