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# Approaches to the σ = ½ phenomenon in multiplicative number theory (i)

by

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## Abstract:

In this note we explain why the line of convergence of $\sum\mu(n)/n^s$ is undecidable in (½, 1) in arithmetic and the conclusion σ = ½ is the only one consistent with reasoning extended from arithmetic reasoning. The Riemann hypothesis follows as a consequence as does the simplicity of the zeros. The method extends to zeta functions which satisfy certain minimal conditions.

## Introduction

The argument here for the zeta phenomenon is very straight forward. We need to be very clear that inductive reasoning is at the very base of sensible reasoning and some time is spent working around this topic as the method should have application to other problems in number theory. Another important part of things is the human capacity to recognise pattern. The formula for the sum of the first N natural numbers is about recognising pattern in the step from k to k+1 in the inductive proof. This has little to do with logical reasoning but is a cognitive quality that many people enjoy. A problem in the following discussion is about trying to recognise non-pattern and how non-pattern may be understood in proof.

Mathematicians use the familiar combination of inductive and deductive reasoning to discuss things. The inductive reasoning is the inner reasoning in which not everything is assumed and it is the recognition of pattern that suggests extension in thought. Deductive reasoning is the outer reasoning which has all the pieces in the argument defined and understood. But the two forms of reasoning are used in tandem to create new entities inductively, which are then used in deductive reasoning.

Throughout this discussion arithmetic will mean Peano arithmetic without the assumption of universal quantification and complete induction. It is the arithmetic, available to anybody using inductive and deductive reasoning within normal classical logic. Complex analysis will then be seen as the argument system and language which includes complete induction and universal quantification in line with ZFC set theory for example, although it is the author's current personal attitude that the axiom of infinity has more to do with set theory than number theory.

The orientation adopted in this explanation is that the language of complex analysis used to discuss the theory of the Riemann zeta function may be developed from arithmetic without the assumption of universal quantification. This is in part a return to Kronecker's arithmetic where we take the view that we make up complex analysis from finite arithmetic using everyday inductive reasoning. In the more formal theory, there is the additional ***assumption*** that it is sensible to write about the entities created in the construction of the real numbers as these are **assumed** things, the existence of which, for the most part, is not verifiable. They are not the results of finite construction. We are however, able to use inductive reasoning to be convinced, using substitution and approximation, that the results about arithmetic which are derived in this wider language cannot be contradicted in arithmetic. We also know from the results of formal logic that there are theorems in complete arithmetic which cannot be proven within that system.

The notions of function, transformation and process are all ideas which allow the concept of inverse. The overview of the discussion is to see aspects of the theory of the Riemann zeta function defining in the language of analysis, a transformation which is non-invertible. We see a move from an inductive state to an equivalent non-inductive state in the language shift from arithmetic to analysis in the theory of the zeta function. This is seen to produce statements which are non-inductive because of their multiplicative foundations, and then non-provable in arithmetic.

An additional '*assertion*' will be introduced which in other places may be viewed as a companion '*axiom*', to the axiom of infinity of formal set theory which provides arithmetic with complete induction.

*If in arithmetic, an unbounded number of thought processes are necessary for proof of a collection of theorems, then there does not exist a provable theorem in arithmetic which establishes the truth of these theorems.*

In line with this we see there is a strong limitation on the global link between the multiplicative numbers and the inductive numbers in arithmetic. That is, we can have an understanding of the inductive numbers finitely (n+1 is the successor of n) but we have to know more and more about multiplicative structure to have a corresponding understanding of multiplicative structure. Elementary proofs of the prime number theorem, Breusch [2] pretty well define the limitations of the local link with the ordered prime numbers. The theorem is local in the sense that it is about the primes less than or equal to x even though it is usually written as $\pi(x) = x/\ln(x) + o\,(x/\ln(x))$ as $x \to \infty$. The theory of the Riemann zeta function shows the natural way to think about the distribution of primes lies in complex analysis since the arithmetic proof that $\pi(x) = x/l(x) + o\,(x/l(x))$ as $x \to \infty$ where $l(x) = 1+1/2+1/3+\ldots+1/[x]$ does not have a natural extension in the language of arithmetic.

We hand over to the increased power of analysis to interpret the prime number distribution in terms of the zeros of the Riemann zeta function. The extension to further terms to describe the distribution of primes is seen to be naturally available in the words of analysis with the natural logarithm, the logarithmic integral and the zeros of the Riemann zeta function. Difficult as elementary proofs of the prime number may be, they are still counting exercises on the prime numbers and the theorem is not at all concerned with the specific prime structure of the natural numbers, other than the simple logical distinction that some are prime numbers and some are not prime numbers.

When we think about the convergence of $\sum\mu(n)/n^s \quad (1 \le n \le \infty)$ we have inbuilt a global property of the Möbius function with the all the values of the function involved in the convergence of the series in a half plane. One back from this, assuming the natural ordering of numbers, the ordering of all the square free numbers has a bearing on the half plane of convergence. From the point of view of analysis there may be ways of getting some information about the convergence but this is not so in arithmetic. We need to know too much about the relationships between prime products to get results which embody this knowledge. The prime factorisations and the order of the square free numbers need to be worked out before we can understand the size of the Möbius sum function in arithmetic. This activity becomes unbounded as $x \to \infty$. We do not have an inductive link in multiplicative structure to shorten this requirement. The theoretical process of estimating the sum function by solving equations leads to a form of circularity in language (see Appendix 2). The global connection from the equation $\sum 1/n^s \quad = \Pi(1\text{-}1/p^s)^{-1}$ defines the limits of connection between the additive and multiplicative structure of the natural numbers. This formula is the gateway in analysis to properties of primes but it presents a wall in arithmetic which cannot be climbed. Although it is not customary in number theory to distinguish between the analytic primes and the arithmetic primes there is a distinction which should be remembered.

## Section 1

## Surety in explanation

The traditional explanation of axiomatic systems for the non- logician is that they produce the working material in independent assumptions so there is clarity in all the logical workings which lead to the proofs of theorems which follow from the assumptions.

A simple split in axiom statements is between working type assumptions which contain the content for the language and the thought type assumptions which are more to do with native reasoning. Peano's axiom for complete induction allows arguments to be closed off with universal quantification.

The view is taken in this discussion that universal quantification in analysis and added to incomplete arithmetic is a powerful devise in language but from a common inductive standpoint '*without exception'* has meaning but 'f*or all'* does not have meaning in the unbounded case. Unbounded inductive argument is accepted but no extra logical processes are involved – we do not gain the capacity for unbounded thought assuming mathematical induction– but we have the capacity to believe that a pattern with inductive structure is imaginable and the capacity to reason that acceptance of this will never lead to contradiction. We return on a number of occasions to the idea that the proofs in analysis must reduce to inductive proofs even though this may not be evident from some of the constructions. There is however, the assumption of existence in the unbounded case – thus $1+1/4 + 1/9 + 1/16 + \ldots = \pi^2/6$ assumes the existence of an entity – a series whose *convergent* appearance defines an entity π. This is certainly not an assumption of finite arithmetic. We are however, encouraged that we are able to prove in analysis that the area of a circle is $\pi r^2$. This is an analytical result because the entity π can only be understood inductively in arithmetic. It is only after the real numbers have been invented that we are able to discover that $1+1/4 + 1/9 + 1/16 + \ldots = \pi^2/6$ and this is thus not a result of arithmetic.

A problem does arise in the construction of the real numbers in that we cannot count them so the collection of them all does not have the simple inductive comprehension of the natural numbers. Through the construction process we are able to order the real numbers but this is at a theoretical level as we cannot understand consecutive real numbers. However, in mathematical activity, when we are concerned with what real numbers do, we are only concerned with processes which we are able to understand inductively through the '*without exception'* interpretation. In this light sensible activity in analysis does not hinge on anything other than inductive reasoning. For example, we can convince ourselves that $1+(1/4) + (1/9) + (1/16) + \ldots = \pi^2/6$, inductively by getting the difference between the two sides smaller and smaller but we may suspect a certain circularity in line with Poincare's objection to complete induction. But it is good circularity because this type of thing is expressing relationships between entities in the new language. We are not looking for proof that we can evaluate π for example. That is, determine the place of π in the ordering of the real numbers.

We see the uses of infinity in analysis as a convenience in the language, conventions in the language which lead to a tidy ways of explaining proofs in analysis using inductive reasoning. We discuss further how it is our simple inductive reasoning in arithmetic which leads to acceptance of analytical results. In other words the inductive assumptions of natural (non-mathematical) thought are all we ever use in doing arithmetic and analysis. Kronecker's famous quotation is about the order of things – the natural numbers help define infinity, not the other way around. But we cannot object to devices being made up if they are useful and interesting. All we need to do is ensure we stay within the realm of reasoning.

Complex analysis is a simple field extension of the real field and so we can engage in number theory in this domain accepting the sensibility of results in the developing language once we are satisfied the arguments have eliminated counter example 'without exception'.

### Arithmetic, analysis and consistency

Arguments in arithmetic as defined are results in the finite realm. The development of properties of these natural numbers is simply rule following within accepted ways of reasoning. It is the availability of natural numbers which is what one needs to do this kind of number theory. From the counting numbers or the natural numbers described in a more formal way we come across the notion of unbounded and order. The ordering is in line with the pattern 1, 1+1, 1+1+1 … and is simply a>c if there exists b such that a+b = c. Unbounded is recognition that there is no limit to the construction 1, 1+1, 1+1+1, …. .
We know from the practical experience of modern mathematics that the assumption of the infinite set has been a good idea. Nevertheless, for some logicians, it is an assumption which cannot be proved and consequently any arguments which contain this assumption may be thought like the inductive mathematical argument p(2), p(3), ….p(n) .. with p(1) missing (or at least non-verifiable). We cannot eliminate the unease of having no more than a system which is internally consistent, or at least, appears to be so, but which we cannot justify any further than this. Add this assumption into arithmetic and we have surely got to come across things we cannot prove without the assumption. The natural numbers of Peano without the need for universal quantification does not attract the unease of the more restricted theory because the development is within finite reasoning which has more to do with classical logic and cognition than concerns about formal systems. We thus proceed on the latter course relying on the consistency of arithmetic without universal qualification and accepting the constructions and theorems of complex analysis are verifiable to any degree of approximation for any specific choices of functions and variable values we choose to examine.

### Arithmetic and analysis (language)

In the process of moving from simple arithmetic to complex analysis we are involved in the process of building new entities which need new words to describe them and a bigger language which becomes a muddle of the old language and all the new words and sentences from the new creations. In this process there is interest in finding out how strong the old language is relative to the new language and this is one of the past times of elementary methods. Thus we see the *elementary methods* of elementary methods in number theory as a distinction in language as much as a language with different logical structure. There is interest in understanding the results is analysis which are provable in arithmetic or which have interpretation in the arithmetic. It becomes clear that there are results in analysis which have interpretation in arithmetic which do not have proof in arithmetic and this is important to this discussion.

The prime number theorem and its extension to Dirichlet's theorem are quite extraordinary in that there are both arithmetical and non-arithmetical proofs but the elementary proofs do not have the strength to get beyond 'first order' expression. If we let $l(N) = 1+(1/2)+(1/3) + \ldots+(1/N)$ we are able to prove $\pi(N) = x/l(N) + o(x/l(N))$ as $N \to \infty$ but there is no natural arithmetical extension to higher order approximation. It is only in analysis and connection with the logarithmic integral that we find a deeper and cleaner explanation for $\pi(x)$. This then is a language problem as much as a logical problem. This apparent failure of the mother language (arithmetic) to talk about itself clearly is one of the drivers for looking at new structures beyond arithmetic. We indicate in the following notes how this crossroads in language in fact contains an explanation for the truth of the Riemann hypothesis.

As things have played out, analysis and arithmetic are competing entities, analysis forever trying to prove superiority with more and more invention but arithmetic as the mother forever holding the upper hand. The tension is perpetual because analysis is a genuine extension of arithmetic in that the language of analysis does not have complete translation from arithmetic. If the theorem in analysis has interpretation/translation in arithmetic there cannot be a contradiction to the analytical result in arithmetic. This follows directly from the interpretation that analysis is also a language extension of arithmetic where the new language is only using construction processes which are understood inductively from the base arithmetic. What we emphasise in discussing the problem of the Riemann hypothesis is the undecidability in arithmetic of certain statements related to the quasi-Riemann hypothesis.

**Logical equivalence**

Logical implication and logical equivalence are used in the logical working of mathematics to understand relationships. Symbolically, we have $P\rightarrow Q$ and $P \equiv Q$ for theorems P and Q.

One level back from this is the situation where P and Q have components in common but there is no simple logical relationship between them.

For example, the theorems $P(12= 2^2.3)$ and $Q(35=5.7)$ may be proved independently of each other and it requires additional components (‘*something added’)* to set up a framework where we prove $P\equiv Q$. In theory at least if we are given the prime numbers less than or equal to N in order we may order all the prime products, allowing for repeated primes, less than or equal to N. If we call this activity, the theorem R(N), as a mathematical activity we have P+R(N) includes Q and Q+R(N) includes P which in this context may be expressed as $P\equiv Q$. What we know determines whether we have implication or equivalence or less connection.

In the activity process R(N) we need to *‘know’* the primes $p_1$, $p_2$, ....$p_{\pi(N)}$ to get the theorem R(N) for the value N. It should be obvious that knowing this set of primes is a minimal requirement for proving the theorem R(N).

Within human mathematical activity there is a thinker doing the thinking and we wish to think in terms of a lower bound count for the elemental thought processes which an individual needs to go through in order to believe an implication in the mathematical context is true? We do not wish to count the lines in a line by line proof but think about counting different entities and connecting thoughts which need to be located before the proof of some equivalence or implication is possible. Here, we use the counting numbers as a component in examining reasoning processes.

An idea which still needs to be defined is that of the *minimum length of a proof* in arithmetic. If we have two theorems P and Q in arithmetic with $P\rightarrow Q$, we say P is stronger than Q if the minimum length of $P\rightarrow Q$ is shorter than the minimum length of $Q \rightarrow P$. In other words something extra is required in arithmetic to get the equivalence.

Then an unbounded sequence of ever strengthening theorems in arithmetic is not provable in arithmetic because the count of a proof becomes unbounded. This is really implicit in the arguments in this note.

**Axiom:** (non-existence):-

*If in arithmetic an unbounded number of thought processes are necessary for proof of a collection of theorems, then there does not exist a provable theorem in arithmetic which establishes the truth of these theorems.*

This is taken to be an observation which cannot be false, an axiom about natural thought. We see it as a complement to incomplete inductive reasoning. It is not quite the negation of inductive reasoning but this assertion is on that side of things and has more to do with understanding infinity as a device as discussed in earlier sections.. We cannot get non-inductive unboundedness out of human thought processes and consequently, any logical mathematical questions in the consistent extension of arithmetic which are able to achieve this are not provable within arithmetic.

**Definition**

An unbounded sequence of theorems in arithmetic p(1), p(2), p(n) … is called non-inductive if a proof of p(1), p(2), ….p(n) necessarily becomes unbounded as $n \to \infty$.

A simple example is the R(N) theorem of the previous section, which is the derivation of the factorisations of 1,2,3,… N. There is no finite collection of inductive mechanisms which allow us to give a finite explanation of the factorisations of an ever increasing set of consecutive natural numbers. The primes need to be determined in order to compute the factorisations and the primes are unbounded.

The idea is that unbounded theorems in arithmetic are unprovable in arithmetic because of the non-existence axiom.

**Unprovability and undecidability within arithmetic (finite set theory)**

There are different understandings of what 'unprovable' and 'undecidable' mean.

The context for these questions in this discussion is simply about the type of explanation still possible for the Riemann hypothesis.

The main drive overall has been to look for a theoretical proof that the hypothesis is true. The view point here is that the hypothesis is posed in a language and there is the implicit hope that there is the possibility of deciding whether the hypothesis is true or false within the language. The notion of 'within the language' allows for building new structures and theories, in much the same way new words appear with new meanings in common languages. There is also the possibility of wider systems of explanation with more axioms and more explanatory power. There would not seem to be an obvious end to the process of abstraction and it has proved to be one of the very fruitful approaches to problem solving and understanding. It should be recognised that there are some limits to the things which are explainable in a language or at least limits on the nature or quality of the explanation. We use the limitations of argument in arithmetic

Let $\underline{\sigma}$ be any number in [½, 1) and let the quasi Riemann hypothesis be the statement that $\underline{\sigma}$ is the least upper bound for the real part of zeros of ζ in the critical strip.

An argument is shaped along the lines that the value of $\underline{\sigma}$ in the quasi-Riemann hypothesis can only be $\underline{\sigma} = ½$ because that result corresponds to the weakest result and any stronger choice leads to contradiction.

This argument does not exclude the possibility of a more abstract proof or abstract visualisation which is acceptable as a proof but such a proof would not appear necessary for a valid explanation. In other words, the hypothesis itself has a relatively simple explanation in language and logic and thought and does not need more abstraction to arrive at an explanation. This is almost the opposite of Fermat's last theorem where a complex abstract proof is accepted but the existence of a proof in arithmetic using roughly the amount of arithmetic available to Fermat is still undecided.

**Some limitations of proof in arithmetic**

**Example 1**

Numerical investigation into the abc conjecture exposes it as something which is possibly true but this could well be the end of matters.

Firstly, we note an analytical result.

The mapping a → a.rad(a) is clearly 1-1 and we have a 'zeta type' analytic relationship

$\sum 1/(a.rad(a))^s = \prod(1+1/p^{2s} + 1/p^{3s} + \ldots\ldots)$ but experience has shown there are limitations to the sorts of understandings which can be developed from these kinds of relationships.

We see the abc conjecture as a problem which cannot be solved positively in arithmetic because it involves unbounded verification. The process of numerical investigation may yield results which may make the possibility of its truth less likely or more likely. A disproof of the abc conjecture would be equivalent to an unbounded sequence of triples $\{a_n, b_n, c_n\}$ such that for some fixed $\epsilon > 0$ we have $\lim \mathrm{Max}\{|a_n|,|b_n|,|c_n|\}/\mathrm{rad}(|a_n b_n c_n|^{(1+\epsilon)}) = \infty$ as $n \to \infty$.

Now this is explicitly about the prime structure of an unbounded number of numbers but the prime structure of a number is only determined by a process of computation.

We may in fact consider the elements $\mathrm{Max}\{|a_n|,|b_n|,|c_n|\}/\mathrm{rad}(|a_n b_n c_n|)$ as 'essentially different' in some sense or other.

It is easier to contemplate this as being an unreasonable thing to ask of the language of arithmetic than it is to believe that a finite proof in arithmetic could release so much 'equivalent' information.

Both the positive proof and the negative rebuttal would ask for more pattern in arithmetic than the language can actually find finitely.

When problems in number theory involve a mix of additive and multiplicative concepts there is a need to sort out problems which actually are asking for an unbounded amount of verifications which are in some sense independent of each other. We only get finite proof when we find sufficient finite pattern.

**Example 2**

To labour these points a little further we note that we cannot 'know' the factorisation' of every natural number and we have to know quite a lot of information to determine the factorisation of a specific number. If we then define the Möbius function through the theoretical multiplicative structure of numbers and ask the question:
Is $M(x) = \sum \mu(n) = o(x^{½+\epsilon})$ as $x \to \infty$ (summation $1 \le n \le x$)? , we are really asking a lot of the language of number theory in arithmetic.

It is not difficult to construct absurdly difficult questions using multiplicative structure which would not be considered worth asking. Less easy to recognise may be the questions which arrive within tidy theories, the answers to which would provide clear cut results rather than provisional results contingent on something or other being true. Such problems are favoured as being provable because of what they yield.

**Example 3**

In analysis we may express M(X) in terms of the zeros of the Riemann zeta function which leads to an explanation of the analytical M(X) in terms of the placement of the zeros in the critical strip. In arithmetic the explanation of the arithmetical M(X) is through the counting function $\pi(X)$ for the prime numbers.

To see this latter assertion it is convenient to dip into the language of analysis to get the arithmetic result. It follows from Braun [1] that each of $\sum 1/p^s$, $\sum 1/(pq)^s$, $\sum 1/(pqr)^s$ ... may be written as polynomials in $\sum 1/p^s$, $\sum 1/p^{2s}$, $\sum 1/p^{3s}$, ...with rational coefficients. For example

$\sum 1/p^s = p_1(s)$ , $\Sigma 1/(pq)^s = ½\{p_1(s)^2 - p_2(s)\}$ and $\Sigma 1/(pqr)^s = 1/6\{p_1(s)^3 - 3p_1(s)p_2(s) + 2p_3(s)\}$ where
$p_K(s) = \sum 1/p^{Ks}$.

Now if we think in terms of the formula for the sum of the coefficients of the product of two Dirichlet series we then see

$M(x) = 1 - P_1(x) + \{1/2P_2(x) – 1/2P_1(\sqrt{x})\} – \{ 1/6\, P_3(x) - 1/2 \sum P_1 (\sqrt{(x/p)} + 1/3P_1(x^{1/3}) \} + \ldots$

where $P_1(x)$ is the prime number counting function and $P_r(x) = \sum P_{(r-1)}(x/p)$ $(r \geq 2)$.

Even the very best theoretical estimates for $P_1(x)$ do not provide much information about M(X). We may observe that the assumption of the Riemann hypothesis in arithmetic does not allow a proof of the Riemann hypothesis in arithmetic from this formula!

Thus in arithmetic we see the evaluation of M(X) is through a very complicated recursive formula based on π(X) which itself is very irregular on local values.

We assert that the fundamental inductive independence of $\sum p^0, \sum p^0q^0, \sum p^0q^0r^0 \ldots$ means that an unbounded number of essentially different things need to be taken into account in arithmetic to get an arithmetic proof which provides $M(X) = O(X^{\Delta})$ as $X \to \infty$ for any $\Delta < 1$, which is possibly why zero free regions for ζ have an asymptotic nature.

In the theory of the Riemann zeta function we gain analytical results which improve our understanding of M(X). Results which cannot be established by the finite exhaustion method of arithmetic but results we are convinced about using our inductive reasoning.

For example we have the analytic result that $M(x) = \sum \mu(n) = \Omega(x^{½-\epsilon})$ as $\to \infty$ but it is doubtful that we could even prove |M(X)| was not bounded in the language of arithmetic. The observation that a leap in the strength of a result is possible in the language of analysis should not be a surprise because the vocabulary is so much stronger. However, when we are discussing results which cannot be established in the language of arithmetic – the unprovables – we must be careful not to assume we can rank unprovables in sensible thought.

## Interpretation of the Riemann hypothesis in arithmetic

In arithmetic we have a meaning for $X^{a/b}$ as P(X) where $P(X)^b \leq X^a < (P(X) + 1)^b$ where a and b are natural numbers with a<b.

The Riemann hypothesis is equivalent to the theorem that $M(x) = \sum \mu(n) = O(x^{½+\epsilon})$ as $x \to \infty$, (summation $1 \leq n \leq x$), for each $\epsilon > 0$ and with suitable adjustment to the ϵ condition we have a statement in arithmetic equivalent to the Riemann hypothesis. The equivalence is not immediately obvious but it is obvious that this result implies the Riemann hypothesis. If the arithmetical condition is met the reciprocal of the zeta function converges as a Dirichlet series for σ > ½ which rules out zeros of ζ in σ > ½.

### Theorem

With summation $1 \leq n \leq x$,

$M(x) = \sum \mu(n) = O(x^{\underline{\sigma}+\epsilon})$ as $x \to \infty$, is undecidable in (½, 1) in arithmetic.

### Proof

In arithmetic we need to recognise that a proof of an estimate for M(x) must involve knowing more than an amount equivalent to the order and the numerical value of the primes less than or equal to x. No matter how large we choose x to be, we will always be faced with a finite collection which we need to order in terms of both the multiplicative structure of a number and its magnitude relative to the other numbers. As we increase the size of x, new structures (products with k distinct primes) come into play and the ordering of these new structures needs to be related to the ordering of the earlier structures. This added dimension lifts the problem of an estimate from a counting problem to a non-inductive problem due to the complex recursive nature of M(X) as discussed in Example 3. The recognition required here is recognition of non-pattern rather than the familiar pattern recognition required in mathematical induction. Note though there is a problem with proving that something is not there so we look for a reason why the assumption of the quasi-Riemann hypothesis with $\underline{\sigma} < 1$ is unprovable in arithmetic and its consistent extension to complex analysis.

We need to realise that if a conjectured analytical result is not compatible with our base arithmetical reasoning then there will be non-trivial implications.

The purpose of the closing discussion is to show that the decidability of the quasi Riemann hypothesis implies as outcome in normal thought which is impossible.

For practical purposes in computation we are just concerned with things matching up to any specified degree of accuracy. We then take the notion of $\epsilon$ -convergence through the definition (for example) that f(x) is $\epsilon$ convergent to l as $x \to a$ if there exists $\delta$ ($=\delta(\epsilon)$) such that $|f(x)-l| < \epsilon$ whenever $0 < |x-a| < \delta$.

We could talk in terms of $\epsilon$ convergence (where we have convergence in the analytical sense) rather than convergence but it would seem a tedious task to develop classical analysis in this language. It is however an assumption in computational methods that analysis is the natural extension of arithmetic and the results of computational methods are to be believed. We believe this because we arrive at the conclusion through inductive and deductive reasoning and this is all we have ever had to go on. We believe for example that in a relationship between entities in analysis we are able to substitute 'convergent' series to establish the validity of the result inductively.

In the analytic extension of analysis as applied to the Möbius sum function we are able to derive the following result:-

Let $M_1(X) = \sum\mu(n)$  $(1 \leq n \leq X)$ where $\mu$ is the Möbius function and let $M_K(X) = \sum M_{(K-1)}(n)$ $(1 \leq n \leq X)$, $K > 1$. Then $M_K(X) = |\Omega_{+-}(X^{K-1+\underline{\sigma}-\epsilon})$ as $X \to \infty$ (Appendix 1).

Note we are dealing here with the analytic Möbius function but within our reasoning it is the analytic extension of the arithmetic Möbius function and there will not be contradiction between results. We are able to prove that $M_K(X) = \Omega_{+-}(X^{K-½-\epsilon})$ as $X \to \infty$ for K = 1,2, 3 ……

On investigation we see these latter analytical results are dependent on the existence of a zero of $\zeta$ on $\sigma = ½$.

We accept this oscillatory result and not too surprisingly, numerical investigation confirms to some extent this 'minimal' oscillatory behaviour.

The preceding discussions about non-inductive theorems and essentially different theorems in arithmetic come into play here.

We need to accept that these results are limited to the language of analysis because the propositions $M_K(X) = \Omega_{+-}(X^{K-½-\epsilon})$ as $X \to \infty$ for K = 1,2, 3 ……are non-inductive and essentially logically different. Although the proof in analysis is by induction they are logically different cases in arithmetic. It is incomprehensible from the language and reasoning of arithmetic that 'all' these propositions are true.

Similarly, $M_K(X) = \Omega_{+-}(X^{K-1+\underline{\sigma}-\epsilon})$ as $X \to \infty$, K = 1, 2, 3 ….. is incomprehensible in arithmetic as a theorem in [½, 1] unless $\underline{\sigma} = 1$. The value $\underline{\sigma} = 1$ we assume is unprovable using an argument of the nature which concludes $\underline{\sigma}$ undecidable in arithmetic in [½, 1) or excluded because there is no sequence of zeros $\{\sigma_n + it_n\}$ with $\lim \sigma_n = 1$.

Our explanation of the truth of the Riemann hypothesis is simply that we cannot accept in arithmetic one unprovable trumping another unprovable. We do not have the inductive reasoning to think about the relative merits of unprovables. It would be nonsense in the analytic extension of thought to locate an unprovable which was logically stronger than a located unprovable. Thus, the language of analysis will not find a $\underline{\sigma} > ½$. Consequently, numerical investigation will not find a zero off $\sigma = ½$.

Intuitively, we also suggest that if the weakest logical case is unprovable then any stronger case is also unprovable.

Further, since $M(x) = O(\sqrt{x})$ as $x \to \infty$ is a stronger statement than $M(x) = O(x^{½+\epsilon})$ as $x \to \infty$ this statement is also undecidable in arithmetic. Since a calculated multiple zero would negate the modified Merten's conjecture, it follows that the calculated zeros will be simple zeros, Odlyzko and Riele [4].

## Appendix 1

### Notation and usage and conventions:

Generally, if $A(x) = \sum a(n)$ $(1 \leq n \leq x)$ then we also write $A_1(x) = A(x)$ and $A_k(x) = \sum A_{(k-1)}(n)$ $(1 \leq n \leq x)$. We call the $A_k(x)$ the higher summation functions. All integrals are 1 to $\infty$.

All Dirichlet series sums are 1 to $\infty$.

All summations of simple number theoretic functions are over natural numbers $\leq x$.

### Two theorems which are assumed in the following discussion:

1. Let $f(s) = \sum a(n)/n^s$ where $A(x) = \sum a(n)$. Then $f(s) = \int A(x)/x^{(s+1)}\, dx$ (the integral representation for Dirichlet series), Titchmarsh [6].
2. If the a(n) are real and eventually of one sign then the function represented by the series has a singularity at the real point on the line of convergence of the series, Titchmarsh [5].

Let $M_1(x) = \sum \mu(n)$ and for $K > 1$, $M_K(x) = \sum M_{(K-1)}(n)$ where $\mu$ is the Möbius function

### Theorem 1

For $K \geq 1$ let $L_K(s) = \sum M_K(n)/n^s$ .

$L_K(s) = 1/(s-1)(s-2)\ldots(s-K)\zeta(s-K) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

### Proof

We note the trivial estimate $M_K(x) = O(x^K)$ as $x \to \infty$.

The proof is by induction. The method may be used to verify the theorem for K= 1.

$L_{(K+1)}(s) = \sum M_{(K+1)(}n)/n^s = s \int \{\sum M_{(K+1)(}n)\}/x^{s+1}\, dx$

$= s \int \{\sum [x]-n+1)M_K(n)\}/x^{s+1}\, dx$

$= s \int \sum M_K(n)/ x^s\, dx - s \int \sum n M_K(n)/ x^{s+1} + p_K(s)$ where $p_K(s)$ is analytic for $\sigma > K+1$.

Thus $L_{(K+1)(}s) = \{(s/(s-1))L_K(s-1)\} - L_K(s-1) + p_K(s)$

$= \{1/(s-1)\} L_K(s-1) + p_K(s)$ and the result follows.

We use this result to derive the main results about the oscillatory behaviour of the Möbius sum function and the higher summation functions.

As in section 1, let $\underline{\sigma}$ be the smallest real number such that $\zeta(s) \neq 0$ for $\sigma > \underline{\sigma}$.

**Theorem 2**

$\mathbf{M}_K(x) = |\Omega_{+-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to \infty$.

**Proof**

Suppose $M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)}$ is eventually of one sign, where A is a non-zero integer.

Then the function defined by the Dirichlet series, $H_K(s) = \sum(M_K(n)+ An^{(K-1+\underline{\sigma}-\epsilon)})/n^s$ has a singularity at the real point on its line of convergence.
It follows from the preceding theorem that
$H_K(s) = 1/[(s-1)(s-2)\ldots(s-K)\zeta(s-K)] + A\zeta(s-K+1-\underline{\sigma}+\epsilon) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma >$ K.

Moving from right to left along the real axis we find the first singularity of $H_K(s)$ at $s-K+1-\underline{\sigma}+\epsilon =$ 1.
i.e $\sigma = K+\underline{\sigma}-\epsilon$.

Since $H_K(s)$ is then analytic for $\sigma > K+\underline{\sigma}-\epsilon$ it follows that $\zeta(s-K)$ is analytic for $\sigma > K+\underline{\sigma}-\epsilon$.
In other words $\zeta(s)$ is analytic for $\sigma > \underline{\sigma}-\epsilon$. This contradicts the choice of $\underline{\sigma}$.

The fact that this is the best possible result in both the positive and negative direction separately also follows from the statement theorem.

We also have the unconditional analytical results that $M_K(x) = \Omega_{+-}(x^{K-1/2-\epsilon})$ as $x \to \infty$ for K=1,2 3…, since $\underline{\sigma} \geq$ ½. It should also be clear that each of the statements of the theorem (K=1 or K=2 or ....) is equivalent to the quasi-Riemann hypothesis.

## Appendix 2: Estimating the Möbius sum function in arithmetic

For convenience we define the Möbius function μ by $\sum\mu(n)[N/n] = 1$ for $N \geq 1$, where summation is $1 \leq n \leq N$ and $[N/n]$ is the largest integer less than or equal to N/n. It is curious that this definition puts one version of the Riemann hypothesis only a few minutes away in terms of a problem description.

$M(x) = o(x)$ as $x \to \infty$ is provable by elementary methods and is equivalent to the prime number theorem, and gets a first order description for a count of the prime numbers in arithmetic.
We examine a pointer towards thinking that stronger estimates for M(x) may not be possible in arithmetic.
If we think in terms of linear equations, the above definitions for the μ function provide a way of calculating the sum function $M(x) = \sum\mu(n)$ without calculating the individual μ(n).

Indeed, with N = 3, 2 and 1 we have

$3\mu(1) + 1\mu(2) + 1\mu(3) = 1 = P(3)$
$2\mu(1) + 1\mu(2) = 1 = P(2)$
$1\mu(1) = 1 = P(1)$.

From the LHS we have $P(3) - 2P(1) = \mu(1)+ \mu(2)+ \mu(3) = M(3)$…………………(1)

Hence, 1 – 2.1 = M(3).

i.e. M(3) = -1.

Further numerical investigation will reveal that this procedure may be continued indefinitely and we use the system of equations

$1 = P(N) = \sum\mu(n)[N/n]$ for $N \geq 1$, to reduce the coefficients of $\mu(N)$, $\mu(N-1)$, $\mu(N-2)$.. to unity.

We end up with an equation of the form

$P(N) - a(2)P(N-1) - a(3)P(N-2) - \ldots - a(N)P(1) = M(N)$ where $a(2), a(3) \ldots$ are integers.

The number of distinct $P(r)$ for $1 \leq r \leq N$ which appear in these equations is in fact about $2[\sqrt{N}]$, corresponding to $[N/1], [N/2], \ldots [N/[\sqrt{N}]$ and $1,2,3 \ldots [\sqrt{N}]$ or thereabouts. See Gelfond and Linnik [2].

To see the pattern of the general reduction it is convenient to use a vector notation.

$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots \mu(N)[N/N], 0, 0, \ldots)$ and
$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots \mu(N), 0, 0, \ldots)$ , where as usual $\mu$ denotes the Möbius function and $M(N)$ denotes the sum $\sum\mu(n)$ with summation $1 \leq n \leq N$.

We use the usual vector addition and scalar multiplication here.

**Theorem 1**

$\underline{M}(N) = \sum\{M(N/n) - M(N/(n+1)\}\underline{P}^*(n)$ with summation $1 \leq n \leq N$.

This is the unique reduction process described above.

**Proof**

We look at the scalar factor associated with $\mu(k)$ in each side of the equation.

On the LHS the scalar factor is 1.

On the RHS the scalar factor is

$\sum[r/k]\{M(N/r)-M(N/(r+1))$ (summation $k \leq r \leq N$)

$= \sum[r/k]M(N/r) - \sum[r/k]\, M(N/(r+1))$ (summation $k \leq r \leq N$)

$= \sum[r/k]M(N/r) - \sum[(r-1)/k]\, M(N/r)$ (summation $k \leq r \leq N$)

$= \sum\{[r/k] - [(r-1)/k]\}M(N/r)$ (summation $k \leq r \leq N$)

$= \sum M(N/tk)$ (summation $1 \leq t$ )

$= 1.$

So the numerical approach outlined in the introduction has $M(N)$ as the final adjustment needed in the step wise reduction of
$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots \mu(N)[N/N], 0, 0, \ldots)$ to

$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots \mu(N), 0, 0, \ldots)$ moving from right to left.

Whereas this provides a systematic way of calculating $M(N)$ there is not a direct theoretical application as we just end up with $M(N)$ is an estimate for $M(N)$.

In other words the solving of the system of equations by this method – in theory – just arrives back at the quantity rather than a new expression from which it may be possible to get a desired estimate.

This circularity gives a clue about why it may be impossible to decide the order of $M(x)$ in the form $M(x) = O(x^{(a+\epsilon)})$ as $x \to \infty$ for $½ \leq a \leq 1$.

In the process of reducing
$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots \mu(N)[N/N], 0, 0, \ldots)$ to

$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots\ldots\ \mu(N), 0, 0, \ldots..)$ we are essentially dealing with vectors of length N but we are restricted in arithmetic to use only $2[\sqrt{N}]$ specific vectors. The distinct numerical values of $[N/n]$ $(1 \le n \le N)$ are the only ones which come into the reduction process. The restriction to about $2[\sqrt{N}]$ of N possible numbers in the equations which we have available is a structural restriction in any numerical manipulation in getting an estimate for M(x). There don't seem to be enough numbers to work with.

The relationship between the Möbius function and the greatest integer function is of such a nature that it does not seem possible to separate them sufficiently to get an improved estimate for the Möbius sum function.

If we sum the equations $\mu(m) \sum\mu(n)[N/nm] = \mu(m)$ $(1\le n \le N , 1\le m \le N)$
with some rearrangement we see
$M(N) - 2M(\sqrt{N}) = - \sum\sum \mu(n)\, \mu(m)[N/nm]$ summation $(1\le n \le \sqrt{N} , 1\le m \le \sqrt{N})$.

This may be written

$M(N) - 2M(\sqrt{N}) = - Ng(\sqrt{N})^2 + \sum\sum \mu(n)\, \mu(m)\{N/nm\}$ summation $(1\le n \le \sqrt{N} , 1\le m \le \sqrt{N})$, where $g(N) = \sum\mu(n)/n$ $(1\le n \le N)$ and $\{N/k\} = N/k - [N/k]$.
This relationship is quite useful for calculating M(N) over large ranges in computer investigation because M(N) is calculated only using the μ values up to $[\sqrt{N}]$.

We also see here that M(N) may be expressed as a complicated mix of the first $[\sqrt{N}]$ μ values and the distinct values of $[N/k]$ $(1\le k \le N)$ which number about $2[\sqrt{N}]$. Also, because of the known oscillatory properties of M(N), there are unbounded times where $\sum\sum \mu(n)\, \mu(m)\{N/nm\}$ summation $(1\le n \le \sqrt{N} , 1\le m \le \sqrt{N})$ is the dominant (positive) term in the equation $M(N) - 2M(\sqrt{N}) = - Ng(\sqrt{N})^2 + \sum\sum \mu(n)\, \mu(m)\{N/nm\}$ summation $(1\le n \le \sqrt{N} , 1\le m \le \sqrt{N})$.

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April 2012