The σ = ½ phenomenon in multiplicative number theory (ii)

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Introduction

This is the first note in this series which is not open ended. That is, the conclusion about the validity of the Riemann hypothesis follows from the assumptions and as such is a free standing document. This delivery form has not been obvious in earlier notes because the discussions were about possibilities, and as is common with some discussion, the intuition of the writer is not necessarily expressed clearly in words. Yet mathematics demands a continuous train of thought to a conclusion before it is accepted and it is this insistence in language which necessarily has to accompany thought in order to gain more than isolated acceptance. It is in this process that the individual recognises there is something tangible which it is possible to defend.

To borrow from the mathematical folklore about Hardy, it has become obvious after many years of thought that the unproven assumptions in this note are obvious.

As usual naive arithmetic means finite arithmetic without the assumption of complete induction and analysis means the language and system of reasoning which assumes complete induction or the assumption of the infinite set as it is expressed in set theory.

The two languages and reasoning systems are related in that we use normal inductive reasoning in arithmetic to be convinced that analysis is a consistent extension of arithmetic. By this we mean that no contradiction in arithmetic can be forced on arithmetic by a result in analysis. That is, we accepted our base arithmetic reasoning as all we have in terms of absolute verification. Also, we take it as a tautology that complete induction cannot be forced on arithmetic by argument in analysis. The Riemann hypothesis is discussed against these natural constraints. We argue that the existence of a zero in the critical strip off the line $\sigma = \frac{1}{2}$ cannot be accepted in arithmetic without forcing complete induction on arithmetic.

The two assumptions in this note are embodied in conjecture 1 and 2 and also in the assumption that $\zeta(s)$ cannot be evaluated exactly in the critical strip.

Conjectures 1 and 2 (the first assumption) mean that the interpretation of the Riemann hypothesis in finite arithmetic is an unprovable hypothesis, and by the second assumption, we mean there does not exist a finite calculation to evaluate $\zeta(s)$ at any point in the critical strip.

The discussion is about proving that an individual starting from naive arithmetic (without the assumption of complete induction) and developing analysis through the 'inductive' argument of the common man will be unable under all circumstances to accept in finite arithmetic (where he/she started off) that a zero in the critical strip off $\sigma = \frac{1}{2}$ could be established by computer or numerical investigation or demonstrated theoretically. That is, the appearance of a zero off the line would produce an inductive gap which is only filled by complete induction.

November 2010

Remember, it is the 'finite' arithmetician who needs to be convinced in finite arithmetic that there is a zero off the line and not the analyst, since the analyst is happy to accept complete induction.

The argument

In the previous note the idea is discussed that the unprovability of the Riemann hypothesis in naive arithmetic implies the truth of the hypothesis in complex analysis. The explanation may be helped by additional clarification of the role that the difference between naïve arithmetic and analysis plays in this problem. The language of analysis admits words for the entities which are non-verifiable in arithmetic but we use normal inductive reasoning to accept analysis as a consistent extension of arithmetic. A focus here is about what can we done in the language and thought of arithmetic where only finite verification and use of incomplete induction are accepted. The idea is to argue that a zero of ζ off the line σ = ½ produces an undecidable state of thought in arithmetic reasoning. The shift in language and thought from 'unprovable' to 'undecidable' is to get distance from the foggy notion of proving something is unprovable. An undecidable statement is a weaker notion than that of an independent axiom because we look for consequences within the system of reasoning we are using, whereas a new axiom produces a new system which is either stronger or weaker than the original system depending on the point of view taken. In other words 'decidable' in this context is used in relationship to a specific system of reasoning. In this discussion the language of arithmetic simply bans words which imply the assumption of a convergent series. That is not to say in any sense 'convergence' is to be disbelieved in arithmetic – it would be a strange argument to put that we cannot clearly explain a notion of convergence through

$1+1/2+1/4+1/8 + \ldots + 1/2^N - 2 = 1/2^N$ and the inductive reasoning of the common man.

This convergence phenomenon in arithmetic is one of the great discoveries in arithmetic.
It is in the difference between complete mathematical induction and normal induction where we find the difference between arithmetic and analysis. Thus for example, the entity π is not defined with quite as much certainty as say 3 because we cannot evaluate it in arithmetic. By this we mean we cannot pinpoint its position in the ordering of the real numbers but we are able to constantly reduce the 'region' in which π allegedly exists. It is not immediately clear this type of 'undecidable' in arithmetic is the same type which we talk about in relationship to the decision making in the critical strip but this should become clear later. Nevertheless, we have an example of the notion of undecidable with irrational numbers. From the point of view of arithmetic, the existence of π is undecidable, but with no doubt at all π has a well defined healthy existence in analysis and plays quite an amazing role in the language. We refer to this type of un-decidability as π – un-decidability. The purpose of 'digging in' to the arithmetic system as distinct from the full power of the analytical language in relationship to the question of the Riemann hypothesis is to try and transfer the notion of un-decidability of RH in arithmetic to decidability in analysis. Thus, we have two clearly defined systems, with rules for the type of words which may be made up and our normal logical reasoning in either system. The two conjectures given below highlight the need to be aware of the system of reasoning which is being assumed.

Let $M(a, X) = 1 - \sum p^{-a} + \sum p^{-a}q^{-a} - \sum p^{-a}q^{-a}r^{-a} + \ldots$ for $a \geq 0$ where p, q, r, .. denote different prime numbers and each summation is over all possibilities less than or equal to X, where as usual the order of primes does not define different possibilities. We note $M(0, X) = M(X)$ is the Möbius sum function.

In arithmetic we have a meaning for $X^{a/b}$ as P(X) where $P(X)^b \leq X^a < (P(X)+1)^b$ and a and b are natural numbers with a<b. We thus have an interpretation of the Riemann hypothesis about the possible 'size' of the Möbius sum function in arithmetic.

Conjecture 1

No choice $M(X) = |\Omega_{+-}(X^{\Delta})$ as $X\to\infty$ with $1> \Delta \geq 0$ leads to contradiction in arithmetic or $M(X) = |\Omega_{+-}(X^{\Delta})$ as $X\to\infty$ is unprovable in arithmetic for $1> \Delta \geq 0$.

Conjecture 2

No choice $M(X) = |O(X^{\Delta})$ as $X\to\infty$ with $1> \Delta \geq 0$ leads to contradiction in arithmetic or $M(X) = |O(X^{\Delta})$ as $X\to\infty$ is unprovable in arithmetic $1> \Delta \geq 0$.

These conjectures are quite radical for the author but very much in keeping with the 'non-inductive' discussions in Braun [1]. It does highlight the difference between the arithmetical Möbius function and the analytical Möbius function which is as much a reality as the difference between the analytical and arithmetic primes as discussed earlier. In analysis we may express M(X) in terms of the zeros of the Riemann zeta function and this leads to an explanation of the analytical M(X) in terms of the placement of the zeros in the critical strip. In arithmetic the explanation of the arithmetic M(X) is through the counting function π(X) for the prime numbers.

To see this latter assertion it is convenient to dip into the language of analysis to get the arithmetic result. It follows from Braun [1] that each of $\sum 1/p^s$, $\sum 1/(pq)^s$, $\sum 1/(pqr)^s$ ... may be written as polynomials in $\sum 1/p^s$, $\sum 1/p^{2s}$, $\sum 1/p^{3s}$, ...with rational coefficients. For example

$\sum 1/p^s = p_1(s)$, $\Sigma 1/(pq)^s = ½ \{p_1(s)^2 - p_2(s)\}$, $\Sigma 1/(pqr)^s = 1/6\{ p_1(s)^3 - 3p_1(s)p_2(s) + 2p_3(s)\}$, and
$\sum 1/(pqrt)^s = 1/24\ p_1(s)^4 - ¼\ p_2(s)^2 - 1/6p_1(s)p_3(s) + 9/24p_4(s)$, ... where $p_K(s) = \sum 1/p^{Ks}$.

If we think in terms of the formula for the sum of the coefficients for the product of two Dirichlet series (see notes) we then have
$M(x) = 1 - P_1(x) + \{1/2P_2(x) - 1/2P_1(\sqrt{x})\} - \{ 1/6\ P_3(x) - 1/2 \sum P_1 (\sqrt{(x/p)} + 1/3P_1(x^{1/3}) \} +$
$\{1/24P_4(X) - 1/4\sum P_1(\sqrt{(X/p^2)})\ \ 1/6P_1(X/p^3) + 9/24\sum P_1(X^{(1/4)})\} + ...$

where $P_1(X)$ is the prime number counting function and $P_r(X) = \sum P_{(r-1)}(X/p)$ $(r\geq 2)$.

The components in this formula may be reasoned out in arithmetic using finite argument – at least in principle.

Even the very best theoretical estimates for $P_1(X)$ do not provide much information about M(X). We may observe that the assumption of the Riemann hypothesis does not allow a proof of the Riemann hypothesis in arithmetic!

Thus in arithmetic we see the evaluation of M(X) is through a very complicated non-inductive recursive formula based on π(X). We assert that the fundamental inductive independence of $\sum p^0$, $\sum p^0q^0$, $\sum p^0q^0r^0$ ... mean that an unbounded number of essentially different things need to be taken into account in arithmetic to get an arithmetic proof in conflict with conjecture 1 or conjecture 2 as discussed in the Braun [2].

We take as true that the activities of complex analysis can be understood as being consistent with our arithmetic using inductive arguments involving numerical approximation in any equation involving non-arithmetic entities. Further, this extension of arithmetic will not produce a result which can be contradicted by numerical investigation because it would cause an unacceptable contradiction in finite arithmetic.

In light of conjectures 1 and 2 which we take to be true, we interpret the existence of a zero on $\sigma = ½$ as exposing a fundamental separation between naïve arithmetic and analysis and also showing the strength of analytic argument.

In order to prove that the existence of a zero on $\sigma = ½$ represents an endpoint in decidability in analysis with respect to a line $\sigma = \Delta$ in the critical strip, we need to demonstrate that the decision making around a zero off the line $\sigma = ½$ would have to go on forever and ever and ever in arithmetic if we wish to accept the result.

Suppose through the normal counting method discussed in Feinstein [1] we find a zero off the line. This would be a proof in analysis that the Riemann was false. Then we have a zero on a line $\sigma = \Delta$ with $\Delta > ½$. Let $s_0 = \Delta + iT_0$ where $\zeta(s_0) = 0$.

To be acceptable in arithmetic and avoid π – un-decidability we need to be able to evaluate Δ and T_0, otherwise our reasoning is in the language and assumptions of analysis. We cannot achieve this in computer calculations because we cannot rule out π – un-decidability even if Δ and T_0 are assumed to be rational. The only evidence acceptable in arithmetic is a finite argument in theory that $\zeta(s_0) = 0$ for a specified s_0. But in the critical strip ζ may be defined by $\zeta(s) = (1-2/2^s)^{(-1)}(1-1/2^s+1/3^s - 1/4s + \ldots)$ and the verification of a zero will always be subject to π un-decidability as this formula cannot be reduced to an exact finite calculation.
In other words the decision required for a line $\sigma = \Delta$ with $\Delta > ½$ would need a finite theoretical derivation that $\zeta(s_0) = 0$, but we cannot calculate $\zeta(s_0)$ exactly because $\zeta(s_0)$ cannot be specified exactly. Thus the indication of a zero off the line through numerical calculation would not decide a line $\sigma = \Delta > ½$ in arithmetic. To push the arithmetical line to the limit, the calculation of the zeros on $\sigma = ½$ also suffers from the problem of π – un-decidability in arithmetic and so we only will even know roughly where they are, even if a zero has rational arguments. Thus in arithmetic we cannot decide on $\sigma = \Delta$ anywhere in $1 > \sigma \geq ½$.

The theoretical existence of a zero on $\sigma = ½$ allows an inductive explanation that (for example) $M(X) = \Omega_{+-}(X^{\Delta-\epsilon})$ as $X \to \infty$ for 0 for any $\epsilon > 0$ and $\Delta \leq ½$. These results provide a continuum of explanations, none of which may be proved in the language of arithmetic using finite verification.

From the point of view in naïve arithmetic it is a minor miracle that $M(X) = \Omega_{+-}(X^{½-\epsilon})$ as $X \to \infty$ is explainable by an inductive argument and the credibility of such argument is strengthened by the numerical evidence which shows M(X) oscillating with exactly this level of amplitude. Although this level of numerical evidence does not prove anything it is very reassuring.

We thus see something about the geometry of the relationship between arithmetic and analysis. A discontinuity in thought leaping from only $\Delta = 0$ being finitely verifiable in arithmetic to the continuum result in [0, ½]. All on the back of the existence of a zero on $\sigma = ½$.

The discovery of a zero on $\sigma = \Delta > ½$ through computer aided calculation would produce another leap in results to $M(X) = \Omega_{+-}(X^{\Delta-\epsilon})$ as $X \to \infty$ in analysis. But this is not an extension in arithmetical reasoning because Δ is undecidable in our base reasoning– it assumes we can exclude π undecidability in computation. The assumption of a zero is essentially building on an undecidable in our base arithmetical reasoning. We can accept zeros on $\sigma = ½$ through a process of inductive reasoning but the acceptance of a zero off the line would necessarily assume complete inductive reasoning. We do not have an inductive link between $\sigma = ½$ and $\sigma = \Delta$.

There is no way in reason that this phenomenon can be forced on naïve arithmetic. In other words, the zero phenomena in $\sigma > ½$ will not be exhibited in numerical investigation because analysis is a consistent extension of arithmetic and will not produce a result which is in conflict with arithmetical reasoning. We cannot confirm complete induction in arithmetic and a zero off the line would do this.

November 2010

Summary of the reasoning:-

A zero off the line by computation is a disproof of RH in analysis.

This would not produce an inductive link to arithmetic because the 'zero' s_0 is π – undecidable and even if it could be evaluated the calculation $\zeta\,(s_0) = 0$ is π – undecidable.

The indication of such a zero through the 'zero phenomenon' thus produces the disconnected results $M(X) = \Omega_{+-}(X^{1/2-\epsilon})$ as $X\to\infty$ and $M(X) = \Omega_{+-}(X^{\Delta-\epsilon})$ as $X\to\infty$ without an inductive link via arithmetic. Analytic results cannot produce a contradiction in arithmetic and so such a fracture in thought in arithmetic is not possible. That is, no such zero will be found.

Website References: https://www.peterbraun.com.au

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[1] Algebra of Number Forms (ii)

[2] The σ=1/2 phenomenon in multiplicative number theory

Feinstein C.
[1] The Riemann Hypothesis is Unprovable. http://arxiv.org/pdf/math/0309367.pdf

Notes:

1. Let a(n), b(n) be the coefficients of Dirichlet series f(s) and g(s) let A(X), B(X) and C(X) denote

∑a(n), ∑b(n) and ∑c(n) where the c(n) are the coefficients of the Dirichlet product f(s)g(s).

It is well known that C(X) =∑a(n)B(X/n) = ∑b(n)A(X/n). This relationship is used in the formula for M(X) in terms of the prime number counting function.

2. π – undecidability should perhaps be defined in terms of numerical estimates for convergent series. In the case 1+1/2+1/4+1/8 + ….. = 2 we are able to be exact about the computation because of the relationship $1+x+x^2+x^3 + \ldots = 1/(1-x)$. We assume that no such hidden reduction occurs for $1-1/2^s + 1/3^s - 1/4^s +$ in the critical strip ….. Thus, π – un-decidability incorporates an algebraic component.

4. The 'shadow' Möbius sum function T(X) with consecutive values:- T(1), T(2) , .. or

1, 0, -1, 0, 1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 3, 2, 1, 0, -1, -2, -3, -2, -1, 0, 1, 2, 3, 4 …... is clearly $O(\sqrt{X})$ as $X \to \infty$. The Riemann hypothesis may be interpreted in terms of the relatively smooth oscillatory behaviour of M(X).

5. The un-decidability of the Merten's conjecture implies that all calculated zeros will be simple.

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