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A recurrence relationship for deriving the formula for the Möbius sum function and an algebraic consequence - Peter Braun

PART 1

Section 1

In the Algebra of number forms [1] we saw that products formed from $\{ p(s), p(2s), p(3s) \ldots \}$ where $p(s) = \sum 1/p^s$ (the sum over prime numbers) could be ordered $\theta_1(s) < \theta_2(s) < \theta_3(s) \ldots$

We used this to prove that each of the series $\sum 1/p^s$, $\sum 1/(pq)^s$, $\sum 1/(pqr)^s$, ….. may be expressed as a linear sum of the $\theta_i(s)$ with rational coefficients.

In this note we provide an alternative way of proving this result which has an interesting application to algebraic identities.

Recall the sets [p], [pq], [p²], [pqr], [p²q], [p³] ... define number forms and each denotes the full collection of numbers with a particular prime factorisation.

Let $P_N = (\sum 1/p^{Ns})$ and $Q_N = \sum 1/(p_1 p_2 \ldots p_N)^s$ for $N \geq 1$ with $Q_0 = 1$, where the summation is over unique prime products counted once.

Proposition 1.

$NQ_N = P_1Q_{(N-1)} - P_2Q_{(N-2)} + P_3Q_{(N-3)} + \ldots\ldots + (-1)^{(N)}P_{(N-1)}Q_1 + (-1)^{(N+1)}P_N$

Proof

Let $\sum 1/[n]^s$ denote normal summation over all numbers of that form, each number included once in the summation. This is not to be confused with the greatest integer function.

For example $\sum 1/[p_1]^s = \sum 1/p^s$ where the RHS summation is over all prime numbers.

Then note

$\sum 1/[p_1]^s \sum 1/[p_2 p_3 \ldots p_N]^s = N\sum 1/[p_1 p_2 p_3 \ldots p_N]^s + \sum 1/[p_1^2 p_3 \ldots p_N]^s$ ... (1)

$\sum 1/[p_1^2 p_3 \ldots p_N]^s = \sum 1/[p_1^2]^s \sum 1/[p_3 \ldots p_N]^s - \sum 1/[p_1^3 p_4 \ldots p_N]^s$ .... (2)

$\sum 1/[p_1^3 p_4 \ldots p_N]^s = \sum 1/[(p_1^3]^s \sum 1/[p_4 \ldots p_N]^s - \sum 1/[p_1^4 p_5 \ldots p_N]^s$ ....(3)

……...

$\sum 1/[p_1^{(N-1)} p_N]^s = \sum 1/[p_1^{(N-1)}]^s \sum 1/[p_N]^s - \sum 1/[p^N]^s$ ....(N-1)

$\sum 1/[p_1^N]^s = \sum 1/[p_1^N]^s$ ....(N)

These equations in the P, Q notation are

$P_1Q_{(N-1)} = NQ_N + \sum 1/[p_1^2 p_3 \ldots p_N]^s$ ....(1*)

$\sum 1/[p_1^2 p_3 \ldots p_N]^s = P_2Q_{(N-2)} - \sum 1/[p_1^3 p_4 \ldots p_N]^s$ ....(2*)

$\sum 1/[p_1^3 p_4 \ldots p_N]^s = P_3Q_{(N-3)} - \sum 1/[p_1^4 p_5 \ldots p_N]^s$ ....(3*)

....

$\sum 1/[p_1^{(N-1)} p_N]^s = P_{(N-1)}Q_1 + \sum 1/[p^N]^s$ ....((N-1)*)

$\sum 1/[p^N]^s = P_N$ ....(N*)

and the proposition follows by collecting appropriate terms.

Section 2

The first few expressions for the Dirichlet series are:

$\sum 1/p^s = p(s)$ .....(1)
$\Sigma 1/(pq)^s = 1/2\ p(s)^2 - 1/2p(2s)$ .....(2)
$\Sigma 1/(pqr)^s = 1/6\ p(s)^3 - 1/2p(s)p(2s) + 1/3p(3s)$ .....(3)
$\sum 1/(pqrt)^s = 1/24\ p(s)^4 - 1/4p^2(s)p(2s) + 1/3p(s)p(3s) + 1/8\ p^2(2s) - 1/4p(4s)$ .....(4)
$\sum 1/(pqrtu)^s = 1/120p^5(s) - 1/12p^3(s)p(2s) + 1/8p(s)p^2(2s) + 1/6p^2(s)p(3s) - 1/4p(s)p(4s) +$
$-1/6p(2s)p(3s) + 1/5p(5s)$ ......(5)
$\sum 1/(pqrtuv)^s = 1/720p^6(s) - 1/48p^4(s)p(2s) + 1/18p^3(s)p(3s) + 1/16p^2(s)p^2(2s) - 1/8p^2(s)p(4)$
$-1/6p(s)p(2s)p(3s) + 1/5p(s)p(5s) - 1/48p^3(2s) + 1/8p(2s)p(4s) + 1/18p^2(3s) - 1/6p(6s)$ ....(6)

where $p(s) = \sum 1/p^s$.
This provides a systematic way of deriving these expressions considerably less onerous than considering the permutations and combinations of products and establishes that the derivation is unboundedly 'legitimate' by obvious inductive process unlike the permutation and combination approach which has more to do with thoughtful memory.

Using the product formula for the sum of the coefficients of a product of Dirichlet series we thus have a formula for the Möbius sum function in terms of the prime number counting function:-

$M(X) = [1 - P_1(X)\} + \{1/2P_2(X) - 1/2P_1(\sqrt{X})\} - \{ 1/6\ P_3(X) - 1/2 \sum P_1 ((X/p^2) + 1/3P_1(X^{1/3}) \} +$
$\{1/24P_4(X) + 1/3P_1(X/p^3) - 1/4P_2(X/p^2) + 1/8\sum P_1(\sqrt{(X/p^2)}) - 1/4\sum P_1(X^{(1/4)})\} + ...$

where $P_1(X)$ is the prime number counting function and $P_r(X) = \sum P_{(r-1)}(X/p)$ $(r \geq 2)$.

This formula exposes the complicated nature of the Möbius sum function in arithmetic.

Each bracketed term is $\Omega(X^{1-\epsilon})$ as $X \to \infty$ and the number of terms is unbounded. Further, the value of later terms is a function of earlier terms.

An order estimate of the form $M(X) = O(X^{\Delta+\epsilon})$ as $X \to \infty$ with fixed $\Delta < 1$ needs to account for cancellation between an unbounded number of terms $F_1(X)$, $F_2(X)$, $F_3(X)$ ..... where $F_N(X)$ is defined recursively through $F_1(X)$, $F_2(X)$, $F_3(X)$ .....$F_{N-1}(X)$ and the base function $F_1(X)$ is the notoriously awkward counting function for prime numbers.

In the question of the order of M(X) we note that the number of bracketed 'terms' ({ }) on the RHS of the formula for M(X) which have significance is unbounded. To establish this we show that

$\sum 1/p_1^s - \sum 1/(p_1p_2)^s + \sum 1/(p_1p_2p_3)^s - ....+(-1)^{(k+1)} \sum 1/(p_1p_2...p_k)^s$ has a singularity at s = 1 for each k. Further each term has a singularity at s= 1.

$\sum 1/p_1^s - \sum 1/(p_1p_2)^s + \sum 1/(p_1p_2p_3)^s - ....+(-1)^{(k+1)} \sum 1/(p_1p_2...p_k)^s$ may be rearranged in the form

$F(s) = a_1P(s)^k + V_2(s)P(s)^{(k-1)} + ...+ V_k(s)$ where $P(s) = \sum 1/p^s$ and each of the V_i is analytic for $\sigma > ½$.

Clearly $\lim_{\sigma \to 1^+} F(\sigma) = \infty$.

In terms of numerical calculation to obtain the values of the primes p_1, p_2 ....p_N we would need a collection of rules which forever uses the information of prior calculation - the sieve of Eratosthenes for example. That is, the n^{th} prime in arithmetic is determined from the first n-1 primes using counting:- $p_n = ERAT_n(p_1,p_2,p_3,...p_{n-1})$ symbolically where $ERAT_n$ denotes the counting method of the sieve.

This is quite a different situation from a collection of rules which allows values of a function from some point on to be calculated without reference to an unbounded number of earlier function values (not to say the earlier values can't be used – just it is not necessary to do so). Note that the formula for the Möbius sum function –

$M(X) - 2M(\sqrt{X}) = -\sum\sum\mu(n)\mu(m)[N/nm]$ (summation $1 \leq n \leq \sqrt{N}$, $1 \leq m \leq \sqrt{N}$) shows we may calculate M(X) from the values $\mu(1), \mu(2), \ldots, \mu([\sqrt{X}])$ but the calculation does involve unbounded prior calculation as X increases.

It is important to remember that it is an essential proven cancellation of terms which would produce an estimate $M(X) = O(X^{\Delta+\epsilon})$ as $X \to\infty$ with fixed $\Delta < 1$ since we could write down a similar formula for [X] (see Wobbly equations) but this is a simple count of numbers less than or equal to X. The influence of prime number structure does not come into the calculation.

Section 3

A systematic way of evaluating the prime sequence from the values of the Möbius function

If we rearrange this equation making $P_1(X)$ ($=\pi(X)$) the subject-

$$P_1(X) = [1 - M(X)\} + \{1/2P_2(X) - 1/2P_1(\sqrt{X})\} - \{ 1/6\, P_3(X) - 1/2 \sum P_1 ((X/p^2) + 1/3P_1(X^{1/3}) \} + \{1/24P_4(X) + 1/3P_1(X/p^3) - 1/4P_2(X/p^2) + 1/8\sum P_1(\sqrt{(X/p^2)}) - 1/4\sum P_1(X^{(1/4)})\} + \ldots$$

we have a formula from which we are able to determine stepwise the prime numbers since whether N is a prime number will be determined from the value of $\mu(N)$, after evaluating all other terms on the RHS of this equation.

Indeed, suppose we are given the values $\mu(1), \mu(2), \mu(3), \mu(4)$, .. and $P_1(1) = 0$.

We easily see that $P_1(2) = 1$ and $P_1(3) = 2$ and $P_1(4) = 2$ from the above formula.

Further suppose we determine the primes up to K by this method.

Then $\pi(K+1) = 1 - M(K+1) +$ (a number which may be calculated).

Hence the value of $\mu(K+1)$ determines whether $\pi(K+1) = \pi(K)$ or $\pi(K+1) = \pi(K)+1$.

If we use the following:

Two ordered sets of numbers A, B are called logically equivalent if each can be derived stepwise from the other and in this case we write $A \equiv B$, then we have shown that

$\{\mu(1), \mu(2), \mu(3) \ldots\} \equiv \{\pi(1), \pi(2), \pi(3) \ldots\}$.

In a sense, knowing the Möbius function is equivalent to knowing the prime number function and hence the prime numbers.

Perhaps then it is not too surprising that two logical equivalences to the Riemann hypothesis are:-

$M(X) = O(X^{½+\epsilon})$ as $X \to\infty$ and $\pi(X) = li(X) + O(X^{½+\epsilon})$ as $X \to\infty$.

PART 2

Section 1 – Algebraic use of proposition 1

In the derivation of proposition 1 we have used the Dirichlet series form so that we have a realm of convergence $\sigma > 1$. If we think in terms of collections of countable identifiable objects rather than convergence the same argument is applicable.

Futher, if we restrict the collection to a finite realm $\{p_1, p_2, \ldots, p_N\}$, the same result also holds.

In the finite case we could replace $1/p_i$ with X_i and s with p if interest is in the algebraic relationships rather than analytical connections.

For example,

From (1)
$X^p \equiv X^p$
From (2)
$2\, X^pY^p \equiv (X^p+Y^p)^2 - (X^{2p} + Y^{2p})$
From (3)
$6X^pY^pZ^p \equiv (X^p+Y^p+Z^p)^3 -3(X^p+Y^p+Z^p) (X^{2p}+Y^{2p}+Z^{2p}) + 2(X^{3p}+Y^{3p}+Z^{3p})$
From (4)
$24X^pY^pZ^pW^p \equiv (X^p+Y^p+Z^p+W^p)^4 + 8(X^p+Y^p+Z^p+W^p) (X^{3p}+Y^{3p}+Z^{3p}+W^{3p})$
$-6 (X^p+Y^p+Z^p+W^p)^2 (X^{2p}+Y^{2p}+Z^{2p}+W^{2p})+ 3(X^{2p}+Y^{2p}+Z^{2p}+W^{2p})^2 -6(X^{4p}+Y^{4p}+Z^{4p}+W^{4p})$
From (5)
$5X^pY^pZ^pW^pU^p \equiv (X^p+Y^p+Z^p+W^p+U^p)^5 -10(X^p+Y^p+Z^p+W^p+U^p)^3(X^{2p}+Y^{2p}+Z^{2p}+W^{2p}+U^{2p})$
$+15(X^p+Y^p+Z^p+W^p+U^p) (X^{2p}+Y^{2p}+Z^{2p}+W^{2p}+U^{2p})^2 +$
$+20(X^p+Y^p+Z^p+W^p+U^p)^2 (X^{3p}+Y^{3p}+Z^{3p}+W^p+U^p) -30(X^p+Y^p+Z^p+W^p+U^p) (X^{4p}+Y^{4p}+Z^{4p}+W^p+U^p)$
$- 20(X^{2p}+Y^{2p}+Z^{2p}+W^{2p}+U^{2p}) (X^{3p}+Y^{3p}+Z^{3p}+W^{3p}+U^{3p}) + 24(X^{5p}+Y^{5p}+Z^{5p}+W^{5p}+U^{5p})$.

So for example, if we have $X^p+Y^p+Z^p = 0$ then from (5) we see
$(X^{3p}+Y^{3p}+Z^{3p}) = 3X^pY^pZ^p$
$2(X^{4p}+Y^{4p}+Z^{4p}) = (X^{2p}+Y^{2p}+Z^{2p})^2$
$6(X^{5p}+Y^{5p}+Z^{5p}) = 5(X^{2p}+Y^{2p}+Z^{2p}) (X^{3p}+Y^{3p}+Z^{3p})$
$6(X^{6p}+Y^{6p}+Z^{6p}) = 3(X^{2p}+Y^{2p}+Z^{2p})(X^{4p}+Y^{4p}+Z^{4p}) + 2(X^{3p}+Y^{3p}+Z^{3p})^2$
$10(X^{7p}+Y^{7p}+Z^{7p}) = 7(X^{2p}+Y^{2p}+Z^{2p})(X^{5p}+Y^{5p}+Z^{5p})$
$6(X^{8p}+Y^{8p}+Z^{8p}) = 2(X^{3p}+Y^{3p}+Z^{3p}) (X^{5p}+Y^{5p}+Z^{5p}) + 3(X^{2p}+Y^{2p}+Z^{2p})(X^{6p}+Y^{6p}+Z^{6p})$
$6(X^{9p}+Y^{9p}+Z^{9p}) = 2(X^{3p}+Y^{3p}+Z^{3p}) (X^{6p}+Y^{6p}+Z^{6p})+ 3(X^{2p}+Y^{2p}+Z^{2p})(X^{7p}+Y^{7p}+Z^{7p})$
$6(X^{10p}+Y^{10p}+Z^{10p}) = 2(X^{3p}+Y^{3p}+Z^{3p}) (X^{7p}+Y^{7p}+Z^{7p})+3(X^{2p}+Y^{2p}+Z^{2p})(X^{8p}+Y^{8p}+Z^{8p})$
This pattern continues since in the recurrence relationship as we set W=U= ...= 0 and so
$Q4 = Q5 = Q6 = ... = 0$.
The recurrence relationship then becomes
$(-1)^{(N+1)}(X^{pN}+Y^{pN}+Z^{pN}) + (-1)^N(X^{(p-1)N}+Y^{(p-1)N}+Z^{(p-1)N}) (X^p+Y^p+Z^p)+$
$(-1)^{N-1}(X^{(p-2)N}+Y^{(p-2)N}+Z^{(p-2)N})\{ 1/2(X^p+Y^p+Z^p)^2 -(X^{2p}+Y^{2p}+Z^{2p})\} +$
$(-1)^{N-2}(X^{(p-3)N}+Y^{(p-3)N}+Z^{(p-3)N})\{1/3(X^{3p}+Y^{3p}+Z^{3p})\}$.
This reduces to the continuing pattern above assuming $X^p+Y^p+Z^p = 0$.

We note that we can now (for example) consider the case
$6(X^K+Y^K+Z^K) = 2(X^{3p}+Y^{3p}+Z^{3p}) (X^{(K-3)p}+Y^{(K-3)p}+Z^{(K-3)p})+3(X^{2p}+Y^{2p}+Z^{2p})(X^{(K-2)p}+Y^{(K-2)p}+Z^{(K-2p)})$
where $K = p^N$ for any natural number N.
The important thing to note here is that the case p=1 does not produce new conditions on X, Y, Z as we let N take the values 1,2,3, ... since $1^N = 1$ but for odd p >1 we have an unbounded number of algebraic conditions to satisfy.
An explanation of FLT would thus follow by demonstrating that these algebraic relationships impose an unbounded number of conditions on X, Y, Z for odd $p > 1$ and
$XYZ \neq 0$. This type of approach would have been available to the originator of the problem as distinct from the assertion that the accepted contemporary approach of proof is necessary for the explanation of this problem. This is simply a naive belief that a certain approach, because it involves the most profound mathematics of the time, must have some sort of hidden importance. As we well know, the passage of time catalogues all great ideas and puts them in their place. Number theory could certainly proceed at least as quickly in the short term if the current impasse on some of the historically outstanding questions were explained though the eyes of the 'innocent'.

References
[1] Algebra of Number Forms (i) https://www.peterbraun.com.au