A representation theorem for Dirichlet series with coefficients constant on forms

by

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The uniqueness of factorisation of natural numbers allows for the definition of number forms through equivalent classes of numbers having essentially the same form. Thus [p] is the class defining the prime form, [pq] the class of products of two distinct prime numbers, and so on.

Let $N = (p_1)^{a_1}(p_2)^{a_2}\ldots(p_n)^{a_n}$ where the p_i are distinct primes and $a_1 \leq a_2 \ldots \leq a_n$.

Let $\upsilon(N) = \sum a_i$

We order forms using index minimisation.

Firstly, $[N] < [M]$ if $\upsilon(N) < \upsilon(M)$.

Next we order the forms [N] which satisfy $\upsilon(N) = K$. We say such forms have index K.

In $[N] = [(p_1)^{a_1}(p_2)^{a_2}\ldots(p_n)^{a_n}]$ with $a_1 \leq a_2 \ldots \leq a_n$ we call $[(p_1)^{a_1}]$, $[(p_2)^{a_2}]$, ….$[(p_n)^{a_n}]$ the ordered components of the form and $a_1, a_2, \ldots, a_n$ the indices of the component forms. The components are counted from left to right.

Now suppose we have $\upsilon(N) = \upsilon(M) = K$, with $N \neq M$. Let $N = (p_1)^{a_1}(p_2)^{a_2}\ldots(p_n)^{a_n}$ and

$M = (q_1)^{b_1}(q_2)^{b_2}\ldots(q_m)^{b_m}$ where $a_1 \leq a_2 \ldots \leq a_n$ and $b_1 \leq b_2 \ldots \leq b_m$.

Let T be the smallest number such that $a_T \neq b_T$. We easily see that such a T exists and T<n and T<m.

Indeed, without loss of generality, suppose $a_1 = b_1, a_2 = b_2, \ldots a_T = b_T$ where T = n and m > n.

Then $b_1+b_2+\ldots b_T = K$ and $b_1+b_2+\ldots+b_{(T+1)} > K$ contradicting $\upsilon(M) = K$.

We now define the ordering by $[N] > [M]$ if $a_{(T+1)} > b_{(T+1)}$ and $[M] > [N]$ if $a_{(T+1)} < b_{(T+1)}$.

Building forms of index K in line with the ordering defined above starts with the product of K primes defining the smallest form. Then moving from left to right in the component construction we place in the form component with the lowest possible index greater than or equal to unity which allows a form of index K to be constructed. And this method applies until the form is constructed.

For example

$[pqrst] < [pqrt^2] < [pqr^3] < [pq^4] < [p^2q^3] < [p^5]$.

We note properties of Dirichlet series whose coefficients are constant on forms.

Proposition 1

Let $f(s) = \sum a(n)/n^s$, $g(s) = \sum b(n)/n^s$ where the coefficients are constant on forms and let γ be any complex number.

Then Dirichlet series $f(s) + \gamma\, g(s)$, $f(s)g(s)$ each have coefficients which are constant on forms.

Proof

Only the product result requires comment.

Let $h(s) = f(s)g(s) = \sum c(n)/n^s$.

Then $c(n) = \sum a(n/g)b(g)$ where summation is over the divisors of n.

Now if n, n′ have the same form then there is a 1-1 correspondence between n/g and n'/g' where g and g′ have the same form.

Since n/g and n'/g' have the same form we see $c(n) = c(n')$.

We have discussed the ordering of forms:-

$[p], [pq], [p^2], [pqr], [pq^2], [p^3], [pqrs], [pqr^2], [pq^3], [p^2q^2], [p^4] \ldots$.

Let $D_1, D_2, D_3, ..$ denote $[p], [pq], [p^2] \ldots$.

Further, let $f_i(s) = \sum c(n)/n^s$ where $n \in D_i$

This $f_i(s)$ notation will be used through the discussion.

We then have: -

Corollary 1

Given a_i $(1 \le i \le N)$ and b_j $(1 \le j \le M)$ there exists c_k and K ((1 ≤ k ≤ K) such that

$\sum a_if_i(s) \ \sum b_jf_j(s) = \sum c_kf_k(s)$

Indeed, the coefficients of $f_i(s)f_j(s)$ are constant on forms and only a finite number of forms are in the product.

Clearly, if the a_i and b_j are rational then the c_k are rational.

For natural numbers $n \ge 1$ and $\sigma > 1$, let $p_n(s) = \sum 1/p^{ns}$ where summation is over all prime numbers and s may be considered a complex variable with $\sigma > 1$.

Let G be the semi-group generated by $\{p_1(s), p_2(s), \ldots\}$ using multiplication of Dirichlet series.

The typical term is $\theta(s) = p_1(s)^{a_1}p_2(s)^{a_2}\ldots p_n(s)^{a_n}$ with $a_i \ge 0$. We now define an ordering on the elements of G in a way not altogether different from the ordering on number forms.

We define $\upsilon(\theta(s)) = \sum ja_j$ as the index of the element $\theta(s)$ and if $\upsilon(\theta(s)) < \upsilon(\varphi(s))$ we define $\theta(s) < \varphi(s)$.

We next define the ordering on elements with constant index K.

Suppose $\theta(s) = p_1(s)^{a_1}p_2(s)^{a_2}\ldots p_n(s)^{a_n}$ and $\varphi(s)= p_1(s)^{b_1}p_2(s)^{b_2}\ldots p_m(s)^{b_m}$, where each series has index K.

Note $a_i \ge 0$ and $b_j \ge 0$.

Suppose without loss of generality that a_1 and b_1 are non-zero.

If $a_1 < b_1$ we define $\varphi(s) < \theta(s)$. If $a_1 = b_1$ we cancel the term and consider the two derived series. Each series is a non- trivial function of s and we may compare a_2 and b_2

$\theta_1(s) = p_2(s)^{a_2}\ldots p_n(s)^{a_n}$ and $\varphi_1(s)= p_2(s)^{b_2}\ldots p_m(s)^{b_m}$ and follow the ordering rule described in the last stage.

Continuing with this method we must arrive at a first point where the leading series $p_u(s)$ derived from $\theta(s)$ and the leading series $p_v(s)$ derived from $\varphi(s)$ are different.

If $u < v$ we define $\theta(s) < \varphi(s)$.

This ordering may also be defined in terms of the minimum index idea discussed in the context of number forms.

In considering $\theta(s)$ for which $\upsilon(\theta(s)) = K$ we define the product $p_1(s)p_1(s) \ldots\ldots p_1(s)$ of these K terms as the smallest element.

At each later stage we form a product from left to right putting in the $p_a(s)$ with the smallest a for which it is possible to construct a new form product $\theta(s)$ with $\upsilon(\theta(s)) = K$. And this applies for each term until the full product is formed.

The notation we use for these ordered products is $\theta_1(s)$, $\theta_2(s)$, ….

With K = 8 we have the ordering and correspondence:-

$p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)$	$\Longleftrightarrow [q_1q_2q_3q_4q_5q_6q_7q_8]$
$p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_2(s)$	$\Longleftrightarrow [q_1q_2q_3q_4q_5q_6(q_7)^2]$
$p_1(s)p_1(s)p_1(s)p_1(s)p_1(s)p_3(s)$	$\Longleftrightarrow [q_1q_2q_3q_4q_5(q_6)^3]$
$p_1(s)p_1(s)p_1(s)p_1(s)p_2(s)p_2(s)$	$\Longleftrightarrow [q_1q_2q_3q_4(q_5)^2(q_6)^2]$
$p_1(s)p_1(s)p_1(s)p_1(s)p_4(s)$	$\Longleftrightarrow [q_1q_2q_3q_4(q_5)^4]$
$p_1(s)p_1(s)p_1(s)p_2(s)p_3(s)$	$\Longleftrightarrow [q_1q_2q_3(q_4)^2(q_5)^3]$
$p_1(s)p_1(s)p_1(s)p_5(s)$	$\Longleftrightarrow q_1q_2q_3(q_4)^5$
$p_1(s)p_1(s)p_2(s)p_2(s)p_2(s)$	$\Longleftrightarrow [q_1q_2\,(q_3)^2(q_4)^2(q_5)^2]$
$p_1(s)p_1(s)p_2(s)p_4(s)$	$\Longleftrightarrow [q_1q_2(q_3)^2(q_4)^4]$
$p_1(s)p_1(s)p_3(s)p_3(s)$	$\Longleftrightarrow [q_1q_2(q_3)^3(q_4)^3]$
$p_1(s)p_1(s)p_6(s)$	$\Longleftrightarrow [q_1q_2(q_3)^6]$
$p_1(s)p_2(s)p_2(s)p_3(s)$	$\Longleftrightarrow [q_1\,(q_2)^2(q_3)^2(q_4)^3]$
$p_1(s)p_1(s)p_5(s)$	$\Longleftrightarrow [q_1\,(q_2)^2(q_3)^5]$
$p_1(s)p_3(s)p_4(s)$	$\Longleftrightarrow [q_1\,(q_2)^3(q_3)^4]$
$p_1(s)p_7(s)$	$\Longleftrightarrow [q_1\,(q_2)^7]$
$p_2(s)p_2(s)p_2(s)p_2(s)$	$\Longleftrightarrow [(q_1)^2(q_2)^2(q_3)^2(q_4)^2]$
$p_2(s)p_2(s)p_4(s)$	$\Longleftrightarrow [(q_1)^2(q_2)^2(q_3)^4]$
$p_2(s)p_3(s)p_3(s)$	$\Longleftrightarrow [(q_1)^2(q_2)^3(q_3)^3]$
$p_2(s)p_6(s)$	$\Longleftrightarrow [(q_1)^2(q_2)^6]$
$p_3(s)p_5(s)$	$\Longleftrightarrow [(q_1)^3(q_2)^5]$
$p_4(s)\,p_4(s)$	$\Longleftrightarrow [(q_1)^4(q_2)^4]$
$p_8(s)$	$\Longleftrightarrow [(q_1)^8]$.

This correspondence exists generally between the polynomials $\theta(s)$ for which $\upsilon(\theta(s)) = K$ and the number forms [n] for which $\upsilon(N) = K$.

For each $p_n(s)$ we substitute a new prime p^n in the corresponding form.

In terms of earlier notation using $\theta_i(s)$ and $f_j(s)$, these series would be

$\theta_{T+1}(s)$, $\theta_{T+2}(s) \ldots$ $\theta_{T+K}(s)$ and $f_{T+1}(s)$, $f_{T+1}(s)$, $\ldots f_{T+K}(s)$.

We note $\theta_{T+1}(s)$, $\theta_{T+2}(s) \ldots$ $\theta_{T+K}(s)$ are Dirichlet series which are constant on forms.

It follows from corollary 1 that each $\theta_i(s)$ is a linear combination of certain $f_i(s)$.

It is relatively easy to prove that the only $f_i(s)$ used to describe $\theta_{T+1}(s)$, $\theta_{T+2}(s)\ldots$ $\theta_{T+K}(s)$ are in fact $f_{T+1}(s)$, $f_{T+2}(s)$, $\ldots f_{T+K}(s)$.

Further the series $\theta_{T+1}(s)$ has $f_{T+1}(s)$ as a component but no other $\theta i(s)$ does, the series $\theta_{T+2}(s)$ has $f_{T+2}(s)$ as a component but none of $f_{T+3}(s)$, $\ldots f_{T+K}(s)$ does, and so on.

We thus have a system of equations:

$\theta_{T+1}(s) = a_{11} f_{T+1}(s) + a_{12} f_{T+2}(s) + \ldots + a_{1T} f_{T+K}(s)$

$\theta_{T+2}(s) = a_{22} f_{T+2}(s) + \ldots + a_{2T} f_{T+K}(s)$

$\theta_{T+K}(s) = a_{TT} f_{T+K}(s)$

where the diagonal coefficients are non-zero.

Inverting, we thus have the basic result

Theorem 1

Each $f_i(s)$ may be written uniquely as a linear combination of the $\theta_j(s)$ over the rational numbers.

We see for example in the case of forms which are square free in their representatives:

$\sum 1/p^s \equiv 1p_1(s)$

$\sum 1/(pq)^s \equiv 1/2 \{p_1(s)^2 - p_2(s)\}$

$\sum 1/(pqr)^s \equiv 1/6\{ p_1(s)^3 - 3p_1(s)p_2(s) + 2p_3(s)\}$

$\sum 1/(pqrt)^s \equiv 1/24 \{(p_1(s)^4 - 6 p_1(s)^2 p_2(s) + 8p_1(s)p_3(s) + 3p^2(2s) - 6p_4(s)\}$, ...

where $p_K(s) = \sum 1/p^{Ks}$.

These identities are not dependent on the p,q,r .. - prime numbers and the corresponding identities apply to any classes restricted to any nominated collection of distinct primes.

See Algebra of Number Forms (ii) for two applications.

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